INTERPRETATION OF MATHEMATICAL FORMULAE

HOUSTON & KENNELLY.

THE INTERPRETATION

OF

MATHEMATICAL FORMULÆ

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NEW YORK

AMERICAN TECHNICAL BOOK CO.

45 VESEY STREET

1898

PREFACE.

It is wonderful how much is capable of being expressed by a mathematical formula, and how little mathematical proficiency is necessary for its interpretation. Nevertheless, it is commonly believed that a long course of mathematical training is essential to an interpretation of such formulæ as are found in ordinary technological text-books. The authors have endeavored, in this little book, to show that this is fallacious, that, on the contrary, a mere knowledge of arithmetic, as a preparatory training to a perusal of this book, will give to the student all the insight that is needed

to understand applied mathematical formulæ. The authors of course do not claim, however, that those who have read this book thereby become expert mathematicians.

PHILADELPHIA, *January*, 1898.

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THE INTERPRETATION OF MATHEMATICAL FORMULÆ.

ALGEBRA.

CHAPTER I.

ADDITION.

Algebra is that branch of mathematics which treats of the properties of numbers and their general relations by means of symbols. It may, therefore, be regarded as a system of generalized arithmetic. The principal operations dealt with in algebra are addition, subtraction, multiplication, division, involution, evolution,

and the solution of the equations involving these operations. We shall proceed to consider these operations in order.

In arithmetic, if we add two numbers together, we obtain their sum; as, for example, when we say that 5, added to 7, gives a total of 12; or, as we may express it, the sum of 5 and 7 is equal to 12; which, again, may be expressed symbolically, $5 + 7 = 12$. Here the sign $(+)$ is called the *plus sign*, and indicates the operation of addition, while the sign $(=)$ is called the *equality sign*, and indicates the condition of equality between the two things which it connects.

The expression

$$5 + 7 = 12 \tag{1}$$

is called an *equation*, in which the left-hand member consists of two terms; namely, 5 and 7, while the right-hand

member consists of a single term; namely, 12. The equation is read thus:

Five plus seven is equal to twelve.

We may extend the terms of an equation to any desirable extent. Thus we may write the following equations which are obviously true,

$$3 + 3 + 6 = 12, \tag{2}$$

or

$$3 + 3 + 6 = 9 + 3, \tag{3}$$

or

$$4 + 3 + 2 + 1 + \tfrac{1}{2} = 10\tfrac{1}{2}. \tag{4}$$

The number of terms on either side of the equation may be any whatever. All that is necessary is that the sum of the terms in the left-hand member should be equal to the sum of the terms in the right-hand member.

The four preceding equations deal with terms which are all simple numbers or definite numerical quantities. We may,

however, extend the same reasoning symbolically. Thus, in the equation

$$a + b = c \tag{5}$$

we have a statement that the sum of two quantities on the left-hand side, which are represented respectively by the letters a and b, is equal to the quantity on the right-hand side, represented by the letter c. If we make a, equal to 5, and b, equal to 7, we reproduce equation (1), and we are compelled to make c, equal to 12. In other words, giving definite numerical values to the terms on the left-hand side of the equation determines the value of the symbol on the right-hand side. There is this difference, however, that in an arithmetical equation such as appears in equation (1),

$$5 + 7 = 12,$$

we are dealing with numbers only, while in the case of a corresponding algebraic equation,

$$a + b = c,$$

we are not restricted to numerical quantities, because although a, b, and c, may stand for the numbers 5, 7, and 12, respectively, and thus reproduce the arithmetical equation, they may also stand for other magnitudes which are not merely numerical. For example, if a man, weighing a pounds, is suspended from a rope, and the man holds in his hand a weight of b pounds, then the total weight which the rope has to sustain is $a + b$ pounds, and we may express this in the form of the equation

$$c = a + b \qquad \text{pounds,} \tag{6}$$

which is the same as the equation

$$a + b = c \qquad \text{pounds.} \tag{7}$$

Thus, if the man weighs 150 pounds, then $a = 150$, and if the weight he carries in his hand is 10 pounds, then $b = 10$, and the total weight on the rope

$$\begin{aligned} c &= 150 + 10 \\ &= 160 \qquad \text{pounds.} \end{aligned}$$

Here the quantities which are considered in the equation are the magnitudes of certain weights, and these magnitudes are treated numerically, through the use of the unit of weight, or a pound, by which they are compared numerically. Consequently, while an arithmetical equation considers equality between numbers, an algebraic equation may consider equality between any kind of magnitude, from the brilliancy of a fixed star to the area of a piece of land. In all cases, however, the same kind of magnitude only can be compared in an equation; so that if the left-hand

member $a + b$, of equation (7) represents two magnitudes expressed in pounds, the right-hand member c, must also represent a magnitude expressed in pounds.

Moreover, in an arithmetical equation, such as

$$5 + 7 = 12,$$

the terms are all fixed in value, and the equation is susceptible of but one interpretation; namely, that the sum of 5 and 7 is equal to 12; but in an algebraic equation,

$$a + b = c,$$

the fact that the terms are symbolical, enables them to represent any desired numerical values; the only requisite condition being that c, shall be equal to the sum of the values of a and b, or that the equality expressed by the equation shall subsist. Thus, we have seen that

if the weight a, of the man suspended by the rope was 150 pounds, and the weight b, which he carried, was 10 pounds, then c, the tension in pounds weight, supported by the rope, is 160 pounds; but the equation would also correctly represent a case in which the man weighed 230 pounds, and the weight he carried 50 pounds; but in this case the total tension on the rope would be

$$230 + 50 = 280 \qquad \text{pounds.}$$

It is evident, therefore, that the algebraic equation,

$$a + b = c,$$

is susceptible of an infinite variety of interpretations, according to the values which are given to the terms, and is, therefore, of a much more general nature

than the simple type of arithmetical equation,

$$5 + 7 = 12,$$

to which it corresponds.

An algebraic equation involving the operation of addition may extend to any number of terms, just as we have seen an arithmetical equation may similarly extend. Thus, suppose that, at a railway station, a truck is loaded with a number of boxes, and that their individual weights are represented by the symbols a, b, c, d, e, f and g, respectively. Then, if we expressed the total weight upon the truck by the letter w, we obviously obtain the equation

$$w = a + b + c + d + e + f + g \text{ pounds. } (8)$$

This is a symbolic method of making the following statement: The total weight upon the truck of the seven boxes with

which it is loaded, is equal to the sum of the weights of all these boxes.

It is evident that equation (8) may represent an infinite variety of arithmetical equations, each of which would be true in its own particular case. We have only to substitute for the symbols a, b, c, d, e, f and g, their proper numerical values, and the value of w, their total weight, then follows from the equation. If the weights of the individual boxes are expressed in pounds, w, will be determined in pounds. If the weights of the individual boxes are expressed in kilogrammes, w, will be determined in kilogrammes, and so on for any unit of weight. Equation (8) is, therefore, an equation connecting gravitational magnitudes, but it is evident that this equation is capable of representing any summation relation between any kind of magnitude whatever. For example, if we

form an equation for the total amount of rainfall in a week, and represent this total by the letter w, where w, is a quantity expressed in inches of water evenly distributed over a level surface; then if a, b, c, d, e, f and g, represent the rainfall in inches, occurring on each of the seven days of the week respectively, the equation (8) is no longer an equation between gravitational magnitudes, but an equation between rainfall magnitudes, and is susceptible again of infinite variety of interpretations. It is evident that the same algebraic equation would be true for any week in the year, but the actual numerical solution of the equation, obtained by substitution in the terms on the right-hand side, would, probably, vary considerably from week to week.

Similarly, if instead of taking the weekly rainfall, we desired to take the monthly

rainfall, we might construct a similar equation of 30 or 31 terms, according to the month selected, and make their sum still equal to w; but the quantity represented by w, would no longer be a weekly rainfall, but a monthly rainfall. Again, we might make an equation of 365 terms, each term representing the rainfall during one of the days of the year, and equate the whole series to w. Here w, would be no longer a weekly rainfall, but a yearly or annual rainfall. A number of the terms on the right-hand side might be numerically equal to zero, representing the fact that on those days there was no rainfall; the other terms would be numerically expressed in inches and fractions of an inch. In such cases, if the letters of the alphabet are not sufficiently numerous to supply all the terms, the letters of the Greek alphabet, or the German alphabet, may be used, or

capital letters may be employed, or suffixes may be used. Thus, equation (8) might be written if desired

$$w = \alpha + \beta + \mathfrak{d} + A + B + a^1 + b^1.$$

Here the first two terms on the right-hand side are the letters Alpha and Beta, of the Greek alphabet, the third term is the letter *d*, of the German alphabet, the fourth and fifth terms are English capitals, while the sixth and seventh terms are *a* prime, and *b* prime. It is not, of course, usual to adopt a heterogeneous symbolism of this character, and, as a rule, letters of the English alphabet, either small, capital, or suffixed, are the most generally employed.

CHAPTER II.

SUBTRACTION.

If we diminish the value of a certain number, say 12, by another similar number, say 5; or, as it is generally called, if we subtract 5 from 12, we obtain the difference 7. This may be expressed in language by the following equation:

Twelve less five is equal to seven.

Or, in numbers,

$$12 - 5 = 7. \qquad (1)$$

Here the sign ($-$) is called the *minus sign*, and indicates that the quantity which follows it is to be subtracted from the quantity with which it is associated.

This equation may be generalized algebraically in the following manner:

$$a - b = c, \tag{2}$$

which makes the following statement: If from the quantity a, we subtract the quantity b, the remainder will be the quantity c.

In the early stages of arithmetical science, subtraction could only be performed when the number to be subtracted was less than the number from which the subtraction was to be made. Thus, it was considered impossible to subtract six from five. Later development, however, showed that by an extension in meaning of the term subtraction, such an operation could be readily performed, if it was considered that the remainder was *negative*, so that the equation

$$5 - 6 = -1$$

represents that if six be taken from five, the remainder is minus one. Or, equation (2) holds good whether b, is greater than, equal to, or less than a. For example, suppose that a, represents a bank account of five hundred dollars, and suppose that b, represents a draft made on the account. If b, is less than a, the remainder c, will be a positive balance in dollars at the bank. For example if $a = \$500$, and $b = \$300$, c, will be \$200, but if b, is greater than a, say \$550, then the account at the bank will be overdrawn, and will leave a negative balance. Thus,

$$500 - 550 = -50,$$

so that the account will owe \$50 to the bank.

Any quantity to which the minus sign is prefixed may, therefore, be considered as a negative quantity, and from this point of view subtraction is only an extension of

the operation of addition, except that, whereas simple addition only adds positive terms, subtraction, included in addition, considers the addition of terms which are both positive and negative.

For example, if we consider the equation of the preceding chapter,

$$c = a + b,$$

where a, is the weight of a man who is suspended from a rope; b, is the weight of an object which he holds in his hand; and c, is the total weight on the rope. Here, b, instead of being a weight, may be a negative weight, or a sustaining power. For example, if he held in his hand a balloon which exerted an ascensional pull of say 50 pounds, then b, would be -50, and if his own weight were 150 pounds,

$$\begin{aligned} c &= 150 - 50 \\ &= 100 \qquad \text{pounds.} \end{aligned}$$

Here b, is a negative quantity. Again, if the balloon exerted an ascensional pull greater than the weight of the man, say 200 pounds, then the weight which the rope has to sustain would be

$$\begin{aligned} c &= 150 - 200 \\ &= -50 \end{aligned}$$

or, c, would be a negative, or ascensional pull, so that the man and rope would be pulled upward. Consequently, in any equation such as

$$x = a + b + c + d,$$

any of the terms a, b, or c, may happen to have negative values, and the addition would then have to be made with these considerations in view. For the negative terms would have to be subtracted from the positive terms, and if they exceeded

the latter, the sum x, would remain negative. Thus, suppose the equation to represent an account at the bank, and that four payments have been made to the account. Then the equation is equivalent to the statement that the total payment is equal to the sum of the separate payments. Suppose, however, that a and b, are payments which have been made to the credit of the account, and that c and d, are negative payments, or payments which have been made at the expense of the account. Then if

$a = \$100$, $b = \$200$, $c = \$300$, and $d = \$400$,

the equation becomes:

$$\begin{aligned} x &= 100 + 200 - 300 - 400 \\ &= 300 - 700 \\ &= -\$400. \end{aligned}$$

So that the total payment is a negative or debit payment of \$400. With this view of negative quantities, it is evident that simple addition and subtraction fall under one set of operations.

CHAPTER III.

MULTIPLICATION.

In arithmetic, when two numbers are multiplied together the result is called their *product.* Thus, if we multiply 5 by 7, we obtain 35, as the product. This is expressed symbolically as follows:

$$5 \times 7 = 35.$$

Here the sign ($\times$) is called the *multiplication sign*, and is read: "*multiplied by*," so that the equation reads:

Five multiplied by seven is equal to thirty-five.

Expressed in algebraic symbols:

$$a \times b = c$$

or, $$c = a \times b.$$

In these equations c, stands for the product of a and b, whatever those symbols may represent. For example, if a, represents in dollars and cents the wages of a laborer per day, and b, is the number of days' work done by the laborer at this pay, then c, is the total wages due to him at the end of b days. Thus, if a = \$1.60 and b = 6 days, then the amount due to the laborer is

$$\begin{aligned} c &= 1.60 \times 6 \\ &= 9.60 \qquad \text{dollars.} \end{aligned}$$

It is evident that the equation

$$c = a \times b$$

is susceptible of an infinite variety of interpretations, according to the values which may be assigned to a and b. It is to be noticed that if a and b, are mere numbers, or numerical quantities, their product is

also a number, but if a and b, are both physical magnitudes, then their product c, will not be the same kind of quantity as either of the components. Thus, suppose that a weight of a pounds, is raised through a vertical distance of b feet, then we know that the product $a \times b$, is equal to the work done in the process of lifting, and is expressed in units of work called *foot-pounds;* one foot-pound being the amount of work required to raise one pound through a vertical distance of one foot against gravitational force. Consequently, this equation would read:

$$a \text{ (pounds)} \times b \text{ (feet)} = c \text{ (foot-pounds)}.$$

Here it will be seen that the kind of magnitude on the right-hand side of the equation, namely, foot-pounds or work, is different from either of the two magni-

tudes appearing on the left-hand side; namely, weight and distance.

The multiplication sign is frequently omitted between two symbols which are to be multiplied together. Thus, the above equation may be written

$$c = a.b.$$

Here the period takes the place of the multiplication sign. Or, we may write

$$c = ab,$$

both multiplication sign and period being omitted; the symbols being merely written after each other, and, since the order of the two quantities to be multiplied is indifferent, we have also

$$c = ba,$$

where the multiplication sign is likewise omitted. The two quantities a and b,

being written side by side indicate that their product is to be taken. Similarly, any number of quantities written in succession indicate that their continued product has to be taken. For example,

$$x = gac$$

is an equation which means that the quantity x, is equal to the product of the three quantities represented by the symbols g, a and c, respectively. This equation could be written:

$$x = g.a.c$$

or, $$x = g \times a \times c.$$

Thus, if

$$g = 2, a = 4, \text{ and } c = 6$$
$$x = 48.$$

It will be evident that in any equation, such as

$$x = a + b - c,$$

in which the three terms on the right-hand side are represented as simple quantities, any of these terms may stand for products. Thus a, may be the product of e and f, or

$$a = ef;$$

b, may be the product of h, k and j, or,

$$b = hkj;$$

while c, may be the product of p, q, r and s, or,

$$c = pqrs.$$

Consequently, such an equation may be written

$$x = ef + hkj - pqrs.$$

If the values of $e, f, h, k, j, p, q, r, s$, are all given, the value of x, can be determined by first multiplying e and f, together, then multiplying h, k and j,

together, then multiplying p, q, r and s, together and, finally, adding the three products so obtained, the last quantity being considered as negative.

In the arithmetical equation

$$63 = 7\times5 + 7\times3 + 7\times1,$$

in which each of the terms on the right-hand side is a product of a number and number 7, it is clear that the same result will be obtained if we add the numbers by which seven is to be multiplied before making the product, or we may write the equation:

$$\begin{aligned} 63 &= 7 \times (5 + 3 + 1) \\ &= 7 \times 9 \\ &= 63. \end{aligned}$$

Here the symbol (   ), called the *brackets* or *parentheses*, indicates that all of the quantities within them are to be operated

upon by the multiplication sign. In other words, the three quantities within the brackets are to be grouped together and considered as a single quantity.

From the equation

$$63 = 7 \times (5 + 3 + 1)$$

we also derive the equation

$$\begin{aligned} 6 &= 35 + 21 + 7 \\ &= 63. \end{aligned}$$

The same rules apply algebraically. Thus, from the equation:

$$x = a\,(b + c + d),$$

which means that the quantity x, is equal to the product of the quantity a, multiplied by the compound quantity which is the sum of b, c and d, we derive the equation

$$x = ab + ac + ad,$$

from which we see that when a factor a, appears outside of the bracket containing several terms, the factor may be considered to apply to each of these terms in succession.

Thus, suppose that a passenger steamer has accommodation for f first-class passengers, s second-class passengers and g steerage passengers, and that the ship makes ten journeys with every berth filled. Then if x, be the total number of people transported in all, in the 10 trips, we have the equation

$$x = 10f + 10g + 10s$$

or,

$$x = 10\,(f + g + s),$$

where the compound term or quantity in the bracket, namely, $f + g + s$, is the total number of passengers in one trip. Similarly, in n, trips where n, is any number,

the total number of passengers carried will be

$$x = n\,(f + g + s).$$

Sometimes a *straight line* or *vinculum* is used to connect a number of terms into a single group. With the aid of the vinculum the last equation would be written

$$x = n \times \overline{f + g + s}.$$

In a similar manner, several compound terms may be associated together into a product. Thus:

$$x = (a + b + c)\,(d + e + f)$$

means that the quantity x, is the product of two compound terms, the first term being the sum of a, b and c, while the second is the sum of d, e and f. Here if

$$A = a + b + c$$

and $$B = d + e + f$$

then, from the equation,

$$x = AB.$$

For example, if an elevator rises through three stages, the first of which is a, feet, the second b, feet, and the third c, feet, and carries three passengers, the first of whom weighs d, pounds, the second e, pounds, and the third f, pounds, then the total work done in lifting the three passengers through the three stages is represented by the above equation.

If $a = 30$ feet, $b = 20$ feet, and $c = 20$ feet, while $d = 150$ pounds, $e = 100$ pounds and $f = 120$ pounds, we have

$$\begin{aligned} x &= (30 + 20 + 20)\ (150 + 100 + 120) \text{ foot-pounds} \\ &= 70 \times 370 \\ &= 25{,}900 \text{ foot-pounds.} \end{aligned}$$

As an example of multiplication in algebra, we may consider the rule for determining the sum of an arithmetical series; *i. e.*, a series whose successive terms have a constant difference, such as the series

	1	3	5	7	9,	etc.
or	8	16	24	32	40,	etc.

This formula for determining the sum of an arithmetical series is

$$S = n\,[2l - (N - 1)\,d\,],$$

where S, is the sum required, N is the number of terms summed, d is the common difference between any pair of successive terms, l, is the last term, and n, is half the number of terms.

Thus, considering the first series

1	3	5	7	9	11:

Here

$$N = 6\,;\; n = 3\,;\; l = 11\,;\; d = 2\,;$$

and

$$\begin{aligned} S &= 3\,[2 \times 11 - (6-1)\,2] \\ &= 3\,[22 - 10] \\ &= 3 \times 12 \\ &= 36. \end{aligned}$$

The sum required is therefore 36, and we find by actual summation that

$$1 + 3 + 5 + 7 + 9 + 11 = 36.$$

CHAPTER IV.

DIVISION.

In arithmetic if one number is divided by another number, the result is called the *quotient*. This, if 15, be divided by 5, the quotient is 3. Or, in symbols,

$$\frac{15}{5} = 3.$$

The sign of division is also employed to represent the operation. Thus,

$$15 \div 5 = 3,$$

where the sign ($\div$) is called the *sign of division* and is read: "*divided by.*"

Similarly in algebra, the equation

$$c = b \div a$$

means that c, is a quantity which is equal to the quotient of the quantity b, divided by the quantity a. This equation might be written either

$$c = \frac{b}{a}$$

or

$$c = b/a.$$

In one case the *division bar* is used to separate the numerator b, from the denominator a, of the fraction $\frac{b}{a}$, while in the second case, the line is written diagonally, or as a *solidus*. Thus, if $a = 10$ and $b = 17$,

$$c = \frac{17}{10} = 1.7.$$

The process of division may be extended to compound terms which are contained in brackets. Thus,

$$x = (a + b - c) \div (d + e)$$

means that the first compound term, which is the sum of $+a$, $+b$ and $-c$, is divided by the sum of the quantities d and e, or

$$x = \frac{a + b - c}{d + e}.$$

If $a + b - c$, be represented by A, and $d + e$, by B, then

$$x = \frac{A}{B} \qquad \text{or } A \div B.$$

The operations of multiplication and division may be readily associated in an equation. For example,

$$x = e\left(\frac{ay + cd + ef}{c + d}\right).$$

Here the quantity x, is stated to be equal to a quantity e, multiplied by a fraction, the numerator of which is the sum of three products; namely, the sum of ay, cd and ef, while the denominator is the sum

of the quantities c and d. If we suppose that the values of a, c, d, e, f and y, are all given, we may proceed to simplify the equation by forming the numerator and expressing it by the quantity A, and then forming the denominator and expressing it by the quantity B. The equation will then be

$$x = e\frac{A}{B}$$

$$= \frac{eA}{B}.$$

It will be found that a great number of formulæ or algebraic rules for the determination of unknown quantities only involve operations of addition, subtraction, multiplication and division. For example, in calculating the expansion of a gas, reference is usually made to the following formula:

$$V_t = V_0 (1 + \alpha t),$$

where V_0, is the volume occupied by a gas at the temperature of melting ice, or zero degrees Centigrade, V_t, is the volume occupied at some other temperature t degrees Centigrade, and α, is a *coefficient* or constant numerical multiplier which is 0.00366. The equation is, therefore, equivalent to the following statement: "The volume of a gas at a temperature $t°$ C. is equal to its volume at zero Centigrade, multiplied by a quantity which is the sum of unity and the product of the temperature $t°$ C. and 0.00366. Thus, if the volume $V_0 = 300$ cubic feet and $t = 15°$ C., then the volume of the gas at 15° C. is

$$\begin{aligned} V_t &= 300\ (1 + 0.00366 \times 15) \\ &= 300\ (1 + 0.0549) \\ &= 300 \times 1.0549 \\ &= 316.47 \text{ or } 316\ 1/2 \text{ cubic feet,} \end{aligned}$$

approximately.

Again, in the discussion of electric circuits, in which two resistances of r_1 ohms, and r_2 ohms, are connected in parallel, then it may be shown that the joint resistance R, of the pair is

$$R = \frac{r_1\, r_2}{r_1 + r_2} \text{ ohms.}$$

Thus, if $r_1 = 5.5$ and $r_2 = 3.7$, then R, the joint resistance, is

$$R = \frac{5.5 \times 3.7}{5.5 + 3.7}$$

$$= \frac{20.35}{9.2}$$

$$= 2.212 \text{ ohms approximately,}$$

or

$$2\,\frac{212}{1000} \text{ ohms.}$$

In the theory of the conduction of heat, the following equation occurs:

$$Q = K\,\frac{t' - t}{d}\,A\,T.$$

Here, Q, stands for the quantity of heat which passes in a given time, T seconds, through a slab of material whose thermal conductivity is K, and having a surface area of A square centimetres, and a thickness of d centimetres, one face of the slab being maintained at a temperature $t'°$ C. and the other at a lower temperature $t°$ C. Then the equation asserts that the flow of heat is equal to the product of the conductivity K, the difference of temperature $t' - t$, between the faces, the surface area A, and the time T, divided by the thickness d. If we suppose a slab of copper whose conductivity $K = 1.03$, the surface area of the slab being $A = 750$ square centimetres, its thickness $d = 2$ centimetres, and the difference of temperature between the faces of the slab $(t' - t) = 5°$ C. then the flow of heat in the time $T = 10$ seconds will be

$$Q = \frac{1.03 \times 5 \times 750 \times 10}{2} = 19{,}312.5$$

thermal units.

It is important to observe that much may be learned from the form of an equation concerning the nature of the quantities with which it deals, without actually computing or solving it. Thus, in the preceding equation, we observe that the flow of heat passing through a slab of uniform material, increases directly with the time T, because the symbol T, appears as a factor, so that if we double the value of T, we necessarily double the value of Q. Again, the value of Q, increases directly with the active area through which the conduction of heat takes place, because A, appears also as a factor in the product. Similarly, the value of Q, increases directly with the difference of temperature t^1-t, between the faces of the slab. On the

contrary, however, it will be noticed that the thicker the slab, or the greater the value of d, the smaller will be the value of Q; or, in other words, that Q, varies inversely with the thickness d.

In the mensuration of solids we find a formula for obtaining the surface of a right cylinder in terms of its radius and height

$$S = 2\pi r(h + r)$$

where S, is the surface of the cylinder in square inches, including base and top; r, is the radius of the cylinder in inches; h, is the height of the cylinder in inches; and π, is the numerical ratio of the circumference of a circle to its diameter, or, approximately, 3.1416.

Consequently, if $r = 2$ and $h = 10$,

$$\begin{aligned} s &= 2 \times 3.1416 \times 2\ (10 + 2) \\ &= 2 \times 3.1416 \times 2 \times 12 \\ &= 150.797 \text{ square inches, approximately.} \end{aligned}$$

The formula for determining the horse-power of a single-cylinder engine from an indicator diagram is as follows:

$$P = \frac{A\,p\,R\,L}{33{,}000} \text{ horse-power.}$$

Here, the horse-power P, exerted upon one end of the piston, is equal to a fraction whose numerator is the continued product of the active area A, of the piston in square inches, the mean effective pressure p, during one cycle or revolution, in pounds per square inch, the number of revolutions per minute R, and L, the length of the stroke in feet and decimals of a foot.

Thus, if a diagram taken from a cylinder shows that the mean pressure at the back of the cylinder is say 25 pounds per square inch, the length of the stroke $L = 3$ feet, the number of revolutions per minute $R = 160$, and the active area of the back

of the piston was 120 square inches, then the horse-power of the back of the cylinder is:

$$P = \frac{120 \times 25 \times 160 \times 3}{33,000}$$

$$= \frac{1,440,000}{33,000}$$

$$= 43.63 \text{ horse-power,}$$

approximately.

It is shown in treatises on the theory of probabilities that if the probabilities of the truth of a statement made by witnesses are respectively p_1, p_2, p_3, etc., the probability of the truth of a statement made by all the witnesses concurrently is

$$P = \frac{p_1 p_2 \ldots\ldots p_n}{p_1 p_2 \ldots\ldots p_n + (1-p_1)(1-p_2)(1-p_3)\ldots\ldots(1-p_n)}.$$

This equation states that the probability that the event actually occurred, is a

fraction, of which the numerator is the product of the probabilities of true declaration of the respective witnesses, and whose denominator is the sum of two compound terms. The first is the product of the probabilities of true declaration as in the numerator, and the second term is the product of each probability subtracted from unity into all the others similarly subtracted. Thus, if the probability that

A's statements are reliable is			3	in	4	or	$\frac{3}{4} = p_1$
that B's	"	"	4	"	5	"	$\frac{4}{5} = p_2$
" C's	"	"	5	"	6	"	$\frac{5}{6} = p_3$
" D's	"	"	6	"	7	"	$\frac{6}{7} = p_4$

then, if A, B, C and D, all independently concur in stating that a certain event happened, the probability that it actually happened, so far as it can be gauged from this circumstance, is

$$P = \frac{\frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7}}{\frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} + (1-\frac{3}{4})(1-\frac{4}{5})(1-\frac{5}{6})(1-\frac{6}{7})}$$

$$= \frac{\frac{360}{840}}{\frac{360}{840} + \frac{1}{840}}$$

$$= \frac{\frac{360}{840}}{\frac{361}{840}}$$

$$= \frac{360}{361}.$$

The probability is, therefore, 360 in 361, or 360 to 1 that the event actually occurred.

If cannon balls are piled in the form of a square pyramid, the number in one edge of the base being n, the total number of balls N, in the completed pile is

$$N = \frac{n}{6}(n+1)(2n+1).$$

Thus, if $n = 18$

$$N = \frac{18}{3}(19)(37)$$
$$= 6 \times 19 \times 37$$
$$= 4{,}218.$$

If a long rod whose weight is very small be loaded with weights m_1, m_2, m_3, etc., at

distances measured from one end of the rod, equal to x_1, x_2, x_3, etc., then it is shown in treatises on statics that the centre of gravity of the loaded rod is situated at a distance x, from the end of the rod, expressed by the formula

$$x = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ \ldots}{m_1 + m_2 + m_3 + \ \ldots}$$

or x, is a fraction whose numerator is the sum of the products formed by each weight into its distance; while the denominator is the sum of the weights or the total weight of the loaded rod.

This is sometimes written

$$x = \frac{\Sigma mx}{\Sigma m},$$

where Σmx means the sum of all terms of the type m, x, and Σm means the sum of all terms of the type m.

Thus if $m_1 = 3$ oz. $\quad x_1 = 5$ inches
$m_2 = 2$ oz. $\quad x_2 = 10$ "
$m_3 = 4$ oz. $\quad x_3 = 12$ "
$m_4 = 1$ oz. $\quad x_4 = 16$ "

Then

$$x = \frac{3\times5 + 2\times10 + 4\times12 + 1\times16}{3+2+4+1}$$

$$= \frac{15 + 20 + 48 + 16}{3+2+4+1}$$

$$= \frac{99}{10} = 9.9 \text{ inches from the end.}$$

It may be shown that the probability p, of dealing all the 13 cards of one suit to one player from a whist pack of 52 shuffled cards among four players is

$$p = \frac{13\,! \times 39\,!}{52\,!}.$$

Here p, the probability is equal to a fraction, the denominator of which is the *factorial* of 52; *i. e.*, the product 52, 51. 50, 49, etc., 5, 4, 3, 2, 1; or the continued

product of all the numbers between that number and unity inclusive, while the numerator is the product of the factorial of 13 and the factorial of 39. A factorial is represented by the note of admiration following the number. In this case

$$p = \frac{1}{635{,}013{,}559{,}600},$$

or one chance in about 635 billions.

CHAPTER V.

INVOLUTION. POWERS.

IF we multiply any number, say 5, by itself, we obtain what is called its *square* or

$$5 \times 5 = 25,$$

so that 25, is the square, or the *second power* of 5.

Similarly, if we multiply a number by itself twice in succession, we obtain what is called its *cube*, or

$$5 \times 5 \times 5 = 125,$$

so that 125, is the cube, or the *third power* of 5.

In the same way

$$5 \times 5 \times 5 \times 5 = 625$$

is the *fourth power* of 5, and $5 \times 5 \ldots n$ times in succession is the *nth power* of 5. A special notation is employed to represent powers briefly and conveniently. Thus

$$5^2 = 25,$$

where the small number appended is called the *index*, or the *exponent*, or the *power* of 5, and the equation is read, 5 squared, or raised to the second power, is equal to 25.

In the same way

$$7^2 = 49, \text{ because } 7 \times 7 = 49.$$

Similarly,

$$5^3 = 125, \text{ because } 5 \times 5 \times 5 = 125,$$

and the equation is read, 5, raised to the third power, or cubed, is equal to 125.

Similarly, $16^3 = 4{,}096$, or 16, raised to the third power is equal to 4,096.

Similarly,

$$5^4 = 625$$

$$5^5 = 3125,$$

and so on for all powers of 5,

or $$5^n = n\text{th power of } 5.$$

This equation may be generalized algebraically as follows:

$$x = a^n = n\text{th power of } a.$$

Thus, if

$$a = 2 \text{ and } n = 3;\ x = 8.$$

If

$$a = 256 \text{ and } n = 2;\ x = 256^2 = 65{,}536.$$

If we multiply two powers together, as, for example, when we multiply

$$5^2 \times 5^3, \text{ or } 25 \times 125,$$

the product is 5^5 or 3,125.

Here the exponent, or *index of the product,* is obtained by adding the exponents or indices 2 and 3,

or $$5^2 \times 5^3 = 5^{(2+3)} = 5^5.$$

This rule is of general application. The index or exponent of a product of two powers is always equal to the sum of their individual indices or exponents. This rule is expressed algebraically as follows:

$$x^a \times x^b \times x^c = x^{(a+b+c)}$$

where x, is any number; a, b, and c, are the powers of x, while their product is x, raised to the sum of those powers. For example,

$$10^1 \times 10^2 \times 10^3 = 10^{(1+2+3)} = 10^6 = 1{,}000{,}000,$$

or, $$10 \times 100 \times 1{,}000 = 1{,}000{,}000.$$

It is evident that

$$10^1 = 10,\ 10^3 = 1{,}000,$$
$$10^2 = 100,\ 10^4 = 10{,}000,$$

and generally

$$10^n = 1 \text{ followed by } n \text{ zeros.}$$

From the law of addition of indices in the formation of products, it follows that

$$10^2 \times 10^0 = 10^{(2+0)} = 10^2$$

so that if we multiply 10^2, or 100 by 10^0, we obtain the same quantity, or 100, but this is what we would obtain if we multiply 10^2 by unity. Consequently, we infer that $10^0 = 1$. This is a general rule, expressed algebraically as follows:

$$x^\circ = 1, \text{ whatever number } x, \text{ may be.}$$

By the foregoing rule of the summation

of indices in the formation of products, it follows that

$$5^3 \times 5^{-3} = 5^{(3-3)} = 5^0 = 1.$$

Consequently, if we multiply 5^3 by 5^{-3} we obtain unity as the product. But if we multiply 5^3 by $\frac{1}{5^3}$; *i. e.*, the reciprocal of 5^3, we obtain unity as the product. Consequently,

$$5^{-3} = \frac{1}{5^3}.$$

This rule is of general application, and

$$10^{-2} = \text{reciprocal of } 10^2 = \frac{1}{10^2} = \frac{1}{100},$$

and, generally,

$$a^{-n} = \frac{1}{a^n}$$

whatever numbers a and n, may be. As

an example of the use of positive and negative indices, we may take the following:

The frequency of oscillation corresponding to the limits of the visible spectrum; *i. e.*, the number of vibrations per second in the ether producing visible light, lie between, approximately, 394,000,000,000,000 and 762,000,000,000,000 double vibrations per second. These may be expressed as

$$3.94 \times 10^{14} \text{ and } 7.62 \times 10^{14};$$

i. e., between 3.94 (10 × 10 . . .) 14 times in all, or 1 followed by 14 zeros, and 7.62 (10 × 10 . . .) 14 times in all, or 1 followed by 14 zeros.

Again, the wave-lengths in free ether corresponding to these frequencies are contained between the limits

0.000076 and 0.00004 centimetres,

which may be expressed 7.6×10^{-5} and 4×10^{-5} centimetres ; *i. e.*,

$$7.6 \times \frac{1}{10^5} = 7.6 \times \frac{1}{100,000} = \frac{7.6}{100,000}$$

centimetres, and

$$4 \times \frac{1}{10^5} = 4 \times \frac{1}{100,000} = \frac{4}{100,000}$$

centimetres.

The following table will still further illustrate the subject of positive and negative indices:

$$10^0 = 1;\ 10^1 = 10;\ 10^2 = 100;\ 10^3 = 1,000,$$

etc.

$$10^{-1} = 0.1;\ 10^{-2} = 0.01;\ 10^{-3} = 0.003,$$

etc.

It is important to observe that the property of the summation of indices in the

formation of the product only applies to powers of the same base. Thus,

$$7^5 \times 7^{-3} = 7^2,$$

because the indices 5, 3 and 2, in this equation are the indices of a common base; *i. e.*, 7; but it would obviously not be true of

$$7^5 \times 5^{-3} = 7^2 \text{ or } 5^2,$$

because here the powers are of different bases; namely, 5 and 7.

If we allow a stone to fall from the hand toward the earth, the formula which expresses the distance, through which it will fall in a given time t seconds, is

$$s = \frac{gt^2}{2}$$

feet.

This equation is equivalent to the statement: The vertical distance in feet

through which the stone will descend after t seconds have elapsed from the moment of release of the stone, will be the product of the quantity g, into the square of the time t seconds, divided by 2; g, is known to be, approximately, 32.2 feet per second per second, so that the equation becomes

$$s = \frac{32.2t^2}{2}$$
$$= 16.1t^2 \qquad \text{feet.}$$

If $t = 1$, $t^2 = 1 \times 1 = 1$ and $s = 16.1$ feet.

If $t = 2$, $t^2 = 2 \times 2 = 4$ and $s = 64.4$ feet.

If $t = 1.5$, $t = 1.5 \times 1.5 = 2.25$ and $s = 36.225$ feet.

Again, the volume of a sphere of radius r feet, is known to be

$$V = \frac{4\pi r^3}{3} \qquad \text{cubic feet;}$$

where V, is the volume in cubic feet and π, is the ratio of the circumference to the diameter, or 3.1416, approximately.

Consequently,

$$V = \frac{4 \times 3.1416\, r^3}{3}$$
$$= 4.1888\, r^3.$$

Thus, if $r = 1$, $V = 4.1888\ (1 \times 1 \times 1) = 4.1888$ cubic feet.

If $r = 2$, $V = 4.1888\ (2 \times 2 \times 2) = 33.5104$ cubic feet.

If $r = 1.75$, $V = 4.1888\ (1.75 \times 1.75 \times 1.75) = 4.1888 \times 5.3594 = 22.45$ cubic feet.

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CHAPTER VI.

EVOLUTION. ROOTS.

WE have seen that *involution* consists in raising a quantity, say a, to some power, or performing the operation

$$x = a^n.$$

Evolution consists in reversing this, or is the inverse of the above operation. Thus, if we know that $5^2 = 25$, or that 25, is the square of 5, we determine by the process of evolution that 5, is the number whose square is 25; 5, is then said to be the *square root* of 25. In the same way, having given the relation by involution,

$$5^3 = 125,$$

evolution shows that the number 5, is the *cube root* or the *third root* of 125. In the same way 2, is the square root of 4, because $2^2 = 4$; 2 is the cube root of 8, because $2^3 = 8$; 3 is the fourth root of 81, because $3^4 = 81$, and so on.

A root of a number is represented symbolically by a radical sign $\sqrt{}$. Thus

$$a = \sqrt[3]{n}$$

means that a, is the cube root of the quantity n, so that

$$a^3 = n.$$

Similarly,

$$a = \sqrt[m]{n}.$$

means that a, is the mth root of the quantity n. The equation

$$a = \sqrt[2]{n},$$

or a, is equal to the square root of n, is often written

$$a = \sqrt{n};$$

that is, the superscript 2, is omitted in the radical sign. Thus, the equation

$$a = \sqrt{64}$$

means that a, is the number whose square is 64, and, consequently $a = 8$. When the expression whose root is to be extracted is a compound term, a *line* or *vinculum* is placed over it, or brackets are employed. Thus,

$$a = \sqrt{\overline{32 + 32}}, \text{ or } a = \sqrt{(32 + 32)}$$

are equivalent to $a = 8$.

As an example in evolution, or the extraction of roots, the following case may be considered. The formula which gives the period of time of complete vibration

of a simple pendulum, making extremely small oscillations, is

$$T = 2\pi \sqrt{\frac{l}{g}} \quad \text{seconds.}$$

This equation is equivalent to the following statement: The time T, occupied by a pendulum in making one complete to-and-fro motion, of indefinitely small amplitude, is the product of 2π, or 6.2832, and the square root of a fraction whose numerator is the length of the pendulum and whose denominator is the intensity of gravity at the location considered. Thus, if the length of the pendulum be 8.05 feet, and g, the intensity or acceleration of gravity, be 32.2 feet per second per second, then

$$T = 2 \times 3.1416 \sqrt{\frac{8.05}{32.2}}$$

$$= 2 \times 3.1416 \sqrt{\frac{1}{4}}$$

$$2 \times 3.1416 \times \tfrac{1}{2}$$

$$= 3.1416 \text{ seconds.}$$

It is evident that

$$\tfrac{1}{2} = \sqrt{\tfrac{1}{4}}$$

because

$$(\tfrac{1}{2})^2 = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}.$$

Again, from a known relation, whose discovery is ascribed to Pythagoras, between the lengths of the sides of a right-angled triangle; namely,

$$\text{Hypothenuse} = \sqrt{(\text{Perpendicular})^2 + (\text{Base})^2},$$

if we have a triangle OAB, Fig. 1, which contains a right angle or 90° at A, then

$$OB = \sqrt{(OA)^2 + (AB)^2}.$$

If $OA = 4$ feet and $AB = 3$ feet

$$\begin{aligned} OB &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

because 5, is evidently the square root of 25, since $5^2 = 25$.

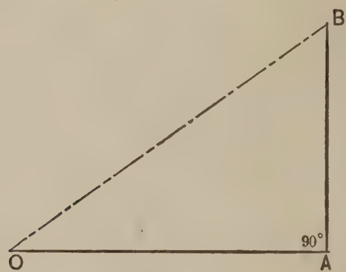

FIG. 1.

Another convenient method of expressing roots or radicals consists in the employment of *fractional indices.* Thus, from the general law of the addition of indices or exponents in the formation of powers, we have:

$$10^{\frac{1}{2}} \times 10^{\frac{1}{2}} = 10^{(\frac{1}{2} + \frac{1}{2})} = 10^1 = 10,$$

so that $10^{\frac{1}{2}}$, multiplied by itself, or squared, gives 10; or,

$$(10^{\frac{1}{2}})^2 = 10, \text{ or } 10^{\frac{1}{2}}, \text{ is the square root of } 10.$$

Similarly,

$$10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}} = 10^{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})} = 10^1 = 10,$$

so that $$10^{\frac{1}{3}} = \sqrt[3]{10}.$$

This is capable of generalization, algebraically, in the formula

$$x^{\frac{1}{n}} = \sqrt[n]{x} = n\text{th root of the number } x.$$

As an example, $9^{\frac{1}{2}} = 3$ because $3 \times 3 = 9$.

We have hitherto considered powers which were formed with indices which are whole numbers or integers, but it is now easy to see what a fractional power means. For example, by the law of the addition of indices,

$$10^{\frac{3}{2}} \times 10^{\frac{3}{2}} = 10^{(\frac{3}{2}+\frac{3}{2})} = 10^3$$

so that

$$10^{\frac{3}{2}} = \sqrt{10^3}$$

or, the square root of the cube of ten;

$$10^{\frac{3}{2}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$$

$$\therefore\ 10^{\frac{3}{2}} = (\sqrt{10})^3;$$

or, the cube of the square root of ten. Consequently,

$$10^{\frac{3}{2}} = (\sqrt{10})^3 = \sqrt{10^3};$$

or, the cube of the square root of ten is equal to the square root of the cube of ten, and generally,

$$a^{\frac{m}{n}} = a^{m(\frac{1}{n})} = \sqrt[n]{a^m}$$

$$= a^{(\frac{1}{n})m} = (\sqrt[n]{a})^m.$$

If then we take a power a^x, and divide its index by some quantity, say 3, we obtain $a^{\frac{x}{3}}$, which is the cube root of a^x, and again

if we multiply its index by any number, say 4, we obtain a^{4x}, which is the fourth power of $a^x = (a^x)^4$.

Again, if we have an equation

$$x = a^{\frac{11}{15}}$$

the equation means that x, is a quantity which is equal to the 15th root of the 11th power of a, or to the 11th power of the 15th root of a; *i. e.*,

$$\begin{aligned} x &= (a^{\frac{1}{15}})^{11} = \sqrt[15]{(a^{11})} \\ &= (\sqrt[15]{a})^{11} = (a^{11})^{\frac{1}{15}}. \end{aligned}$$

CHAPTER VII.

EQUATIONS.

An *equation* is an algebraic expression of equality between two quantities, employing the sign =. Equations differ almost infinitely in nature, complexity, and length. It is impossible, within the limits of this little work, to devote sufficient space to the subject of the treatment or manipulations of equations, so that the solution of any given equation may be obtained. It will be sufficient if the student obtains from this book a clear understanding of the meaning of the statement contained in any equation, so as to be able to interpret its signification. There are, however, a few general and simple

rules which may be set down as a guide for the student in dealing with equations.

Equations may be divided into the following classes:

(1) *Simple equations*, or *equations of the first degree*, are those which involve the first power of an unknown quantity. Thus,

$$x = a$$
$$ax + b = c + d$$
$$x = a\left(p + q + \frac{r}{s}\right)$$

are forms of simple equations, because the unknown quantity represented by x, does not appear except in the first power; *i. e.*, there are no powers of x of the type x^n and no roots of x, of the type $\sqrt{x}$.

(2) *Quadratic equations*, or *equations of the second degree*, are those in which occur the second power of an unknown quantity.

Thus

$$x^2 = a$$
$$ax^2 + bx + c = 0$$

are examples of quadratic equations.

(3) *Equations of the third degree*, or those involving third powers of an unknown quantity: Thus,

$$x^3 = a$$
$$ax^3 + bx^2 + cx + d = 0$$
$$x^2 = ax - \frac{bx^3}{c} + \frac{cx^2}{d} + e$$

are examples of equations of the third degree.

Similarly, equations may be formed of any degree. Equations of the first and second may be solved by definite rules; many of those of the third degree may be solved; but equations of the fourth, or higher degrees, can only be solved rigor-

ously in special cases, although arithmetical approximations to their solution can in all cases be obtained to any desired degree of accuracy.

If the same algebraic operation be performed upon both sides or members of an equation, the equality remains unaltered, although the form of the equation may be greatly changed. For example, if

$$x = a$$

then

$$x + b = a + b$$

because a certain quantity b, is added to both x and a, and if these latter are equal, they must remain equal when increased by the quantity b.

Again $\quad x^2 = a^2$ if $x = a$

or $\quad (x + b)^2 = (x + a)^2$

or $\quad \sqrt{x} = \sqrt{a}$

or $\quad mx = ma$

or $$\frac{x}{n} = \frac{a}{n}.$$

In all these cases, the same operation is performed on each side of the equation. Many transformations may be effected by this process. For example, in the equation

$$x + a = 10,$$

if we subtract the quantity a, from each side of the equation, we obtain

$$x + a - a = 10 - a.$$

On the left-hand side we now have a, added to x, and then a subtracted from the result. Consequently, the *a's cancel*, or may be removed from the left-hand side, and the equation becomes

$$x = 10 - a.$$

We thus see that a quantity may be shifted from one side of an equation to the other

by changing its sign, because, in the original equation a, appeared on the left-hand side under the positive sign, while in the transformed equation it appears on the right-hand side, under the negative sign.

As an example of algebraic equations employed in physics, we may take the following:

If V_o, be the volume of a liquid at temperature 0° C., and V_t, be the volume at a temperature t° C.,

$$V_t = V_o (1 + \alpha t + \beta t^2 + \gamma t^3).$$

The equation states that the volume at temperature t° C., is equal to the volume at 0° C., multiplied by a compound term; *i. e.*, the term within the brackets. This term is the sum of unity, and α times the temperature elevation t, β times the square of the temperature elevation, and γ times the cube of the temperature elevation.

For example, if $V_0 = 100$ cubic centimetres, $t = 10^{\circ}$ C.; $\alpha = 0.00000253$, $\beta = 0.0000008389$, $\gamma = 0.00000007173$, or $\alpha = 2.53 \times 10^{-6}$, $\beta = 8.389 \times 10^{-7}$, $\gamma = 7.173 \times 10^{-8}$.

Then

$$
\begin{aligned}
V_t = {} & 100\,(1 + 10 \times 2.53 \times 10^{-6} + 10^2 \times 8.389 \times 10^{-7} + 10^3 \times 7.173 \times 10^{-8}) \\
& 100\,(1 + 2.53 \times 10^{-5} + 8.389 \times 10^{-5} + 7.173 \times 10^{-5}) \\
& 100\,(1 + 0.0000253 + 0.00008389 + 0.00007173) \\
& 100\,(1 + 0.00018092) \\
& 100\,(1.00018092) \\
& 100.018092 \text{ cubic centimetres.}
\end{aligned}
$$

As an example of an equation involving an infinite series, the following may be considered:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \textit{ad inf.}$$

This shews that the square of the quantity π, which as far as four decimal places is 3.1416, divided by 6, is equal to the sum of the reciprocals of the squares of the successive natural numbers carried to infinity. It is evident that the value of π, might be computed by the aid of such a series. It could never be determined with absolute accuracy because an infinite number of terms would be required, but it could be computed to any desired number of decimal places.

CHAPTER VIII.

LOGARITHMS.

In the equation

$$5^2 = 25$$

the power 2, to which the base 5, is raised in order to be equal to 25, is called the *logarithm* of 25, to the *base* 5. Similarly,

$$a^n = x$$

may be stated by saying that n, is the logarithm of the number x, to the base a; a, may be any positive number, and n, may be any number, positive or negative, and integral or fractional; *i. e.*, a whole number or a fraction, or a whole number and a fraction.

In applied mathematics, logarithms are of considerable importance in performing calculations, as well as in theoretical discussions. The base which is usually selected is 10, and logarithms to this base are called *common logarithms.*

From the general law of the addition of exponents in forming a product, we know that

$$10^2 \times 10^3 = 10^{(2+3)} = 10^5.$$

On the left-hand side of this equation the logarithms of the two factors or quantities to be multiplied are, respectively, 2 and 3, while the logarithm of the product is their sum, 5. This rule is of general application, and may be expressed as follows:

If we sum or add the logarithms of two numbers, we obtain the logarithm of their product.

Thus the logarithm of $100 = 2$, because $10^2 = 100$; the logarithm of $1,000 = 3$, because $10^3 = 1,000$; and the logarithm of $100 \times 1,000 = 5$, because $10^2 \times 10^3 = 10^5$.

If we suppose that a table of logarithms to the base 10, is prepared for all numbers between 0, and infinity, not merely for whole numbers, but for all decimal parts as well, we obtain what is called a *table of logarithms.* All numbers which lie between 1 and 10, will have a logarithm lying between 0 and 1, because $10^0 = 1$ and $10^1 = 10$.

Similarly, all numbers lying between 10 and 100, will have a logarithm lying between 1 and 2, because $10^1 = 10$, and $10^2 = 100$. Similarly, any number lying between 10^a and 10^{a+1}, will have a logarithm lying between a and $a + 1$.

If we take up a table of common logarithms and look for the logarithm of a

number, say 15, we find the decimals .1760913. This represents the *decimal part*, or *mantissa*, of the logarithm, and the *characteristic*, or *whole number*, is to be supplied by the student. Since 15, lies between 10 and 100, its logarithm lies between 1 and 2, or is 1 and some fraction; the decimal fraction being .1760913, the *complete logarithm* is 1.1760913, which may be stated symbolically as follows:

$$10^{1.1760913} = 15,$$

or,

$$\log_{10} 15 = 1.1760913.$$

Here the subscript 10, denotes that 10, is the base employed. Strictly speaking, an indefinitely great number of decimal places in the fractional part or mantissa would have to be employed to obtain the absolutely true logarithm, but for all practical purposes it is found that seven places of

decimals are sufficient, and in many cases even five places are employed for the ordinary degree of accuracy required. We have seen that the logarithm of 15, is 1.1760913, but the logarithm of 150, will only differ from the logarithm of 15, in the characteristic or whole number. This characteristic must lie between 2 and 3, which are the logarithms of 100 and 1,000, respectively, so that the complete logarithm of 150 is 2.1760913.

This may be obtained in another way since

$$15 = 10^{1.1760913}$$

and

$$10 = 10^1$$

$$10 \times 15 = 10^1 \times 10^{1.1760913} = 10^{(2.1760913)}.$$

In the same way the logarithm of 1,500, or

$$\log 1{,}500 = 3.1760913,$$

$$\log 15{,}000 = 4.1760913,$$
$$\log 1.5 = 0.1760913,$$
$$\log 0.15 = \bar{1}.1760913,$$

or has a characteristic of -1

$$\log 0.015 = \bar{2}.1760913, \text{ etc.}$$

Similarly, the logarithm of the number 16, is found from the tables to be .2041200 for the mantissa, and the characteristic is 1, because 16 lies between 10 and 100; therefore, the complete logarithm is 1.2041200, or $16 = 10^{1.2041200}$.

If now we multiply 15 by 16, or make

$$x = 10^{1.1760913} \times 10^{1.2041200},$$

we obtain as their product

$$x = 10^{(1.1760913\ +\ 1.2041200)} = 10^{(2.3802113)}.$$

Here x, is the number whose logarithm 2.3802113 is greater than 2 and less than

3. Consequently, x, lies between $10^2 = 100$, and $10^3 = 1{,}000$. Entering the table of logarithms for the number whose mantissa is .3802113, we find that this corresponds to 24, so that the answer is 240, or

$$15 \times 16 = 240.$$

Here we have performed the computation without the aid of arithmetical multiplication, by the summation of logarithms. This rule is of general application. If we desire to form a product of two or more numbers, we find their logarithms and add them together; the sum is the logarithm of the product sought. As an example of the application of logarithms, we may consider the following:

What is the number of feet in the circumference of the earth considered as a sphere of diameter 7,918 miles of 5,280 feet

each? Here the circumference may be written

$$C = \pi \times 7{,}918 \times 5{,}280,$$

or, taking π as 3.1416,

$$C = 3.1416 \times 7{,}918 \times 5{,}280.$$

If we obtain the logarithms of 3.1416, 7,918 and 5,280, and add them together, their sum will be the logarithm of the quantity C, required. The logarithm of 3.1416 will lie between 0 and 1, or will have a characteristic of 0. The logarithm of 7,918, will lie between 3 and 4, or will have a characteristic of 3. The logarithm of 5,280, will lie between 3 and 4, or will have a characteristic of 3. Consequently, we find by reference to a table of logarithms

$$\begin{array}{lr} \log 3.1416 = & 0.4971509 \\ \text{“} \quad 7{,}918 = & 3.8986155 \\ \text{“} \quad 5{,}280 = & \underline{3.7226339} \\ & 8.1184003\,. \end{array}$$

Consequently

$$C = 10^{8.1184003}$$

and C, lies between 10^8 and 10^9. The number corresponding to the mantissa 0.1184003 is found from the tables to be 1.31341, so that

$$C = 1.31341 \times 10^8 \text{ or } 131,341,000,$$

as far as six places of decimals. If we make the computation we find

$$3.1416 \times 7,918 \times 5,280 = 131,340,996,864,$$

so that the arithmetically computed result differs from the result obtained from seven-place logarithms by less than 4 parts in 131 millions.

If now it should be required to divide 16 by 15, or to compute by logarithms

$$x = \frac{16}{15},$$

we have already seen that the logarithm of 16, is 1.2041200, and of 15, is 1.1760913. Consequently,

$$
\begin{aligned}
x &= \frac{10^{1.2041200}}{10^{1.1760913}} \\
&= 10^{1.2041200} \times 10^{-1.1760913} \\
&= 10^{(1.2041200 - 1.1760913)} \\
& \quad 10^{(.0280287)}.
\end{aligned}
$$

Here x is a number lying between 1 and 10, because the logarithm has a characteristic lying between 0 and 1, and the mantissa 0.0280287 is found in the tables to correspond to the number 1.06667. The computed quotient is found to be 1.0666666; or, as this is sometimes written, $1.0\dot{6}$.

As a further example of the use of logarithms, consider the following problem. The force of gravitation between two homogeneous spheres of masses m_1 and m_2,

grammes, respectively, is expressed by the equation,

$$F = a\frac{m_1 m_2}{d^2},$$

where d, is the distance between the centres of the spheres in centimetres, and a, the gravitation constant; a, is known to be 6.48×10^{-8}, approximately. What then will be the attractive force between the earth and the moon expressed in dynes, if the earth's mass is 6.02×10^{27} grammes, the moon's mass is 7.525×10^{25} grammes, and the mean distance between their centres 3.8444×10^{10} centimetres? The preceding equation may, therefore, be written:

$$F = 6.48 \times 10^{-8} \times \frac{6.02 \times 10^{27} \times 7.525 \times 10^{25}}{3.8444 \times 10^{10} \times 3.8444 \times 10^{10}}$$

$$= \frac{6.48 \times 6.02 \times 7.525 \times 10^{44}}{3.8444 \times 3.8444 \times 10^{20}}$$

$$\frac{6.48 \times 6.02 \times 7.525 \times 10^{24}}{3.8444 \times 3.8444}.$$

Here the logarithms of the three quantities in the numerator, and also of the two quantities in the denominator, have all a characteristic of zero, since the numbers lie between 1 and 10. Consequently, by reference to tables

$\log\ 6.48 = 0.8115750$	$\log 3.8444 = 0.5848286$
$\log\ 6.02 = 0.7795965$	$\log 3.8444 = 0.5848286$
$\log 7.525 = 0.8765065$	1.1696572
2.4676780	

The numerator of the fraction is, therefore, $10^{2.4676780}$, while the denominator is $10^{1.1696572}$. Dividing the numerator by the denominator, or subtracting the logarithm of the denominator from the logarithm of the numerator, we have

$$\begin{array}{r} 2.4676780 \\ 1.1696572 \\ \hline 1.2980208, \end{array}$$

which is the logarithm of 19.8619. Consequently,

$$F = 19.8619 \times 10^{24}$$
$$= 1.98619 \times 10^{25} \qquad \text{dynes.}$$

Another important use of logarithms is in involution and evolution. For example, suppose that it be required to find the cube of 5,280, corresponding to the number of cubic feet in a cubic mile. This will be equivalent to solving by logarithms the equation:

$$x = 5{,}280^3.$$

If

$$5{,}280 = 10^a,$$

then

$$x = (10^a)^3 = 10^{3a}.$$

Here a, is the logarithm of 5,280, and $3a$, is three times this logarithm, as indeed is evident from the fact that the operation of cubing is multiplication to the number by itself twice in succession, or equivalent of

adding its logarithm to itself twice in succession. The logarithm of 5,280, will lie between 3 and 4, and is found from a table to be

$$3.7226339.$$

Multiplying this by 3, we obtain

$$11.1679017,$$

or

$$x = 10^{11.1679017} = 10^{11} \times 10^{0.1679017},$$

and the number corresponding to the mantissa, .1679017, is

$$1.47198.$$

Consequently,

$$x = 1.47198 \times 10^{11} = 147{,}198{,}000{,}000.$$

The true answer is

$$x = 1.47197952 \times 10^{11} = 147{,}197{,}952{,}000.$$

In the same way, if it should be required to determine the 15th root of the number 576, or to compute by logarithms,

$$x = 576^{\frac{1}{15}},$$

we obtain the logarithm of 576, and then divide this by 15.

$$\log 576 = 2.7604225$$

$$\log x = \frac{2.7604225}{15} = 0.1840281.$$

which mantissa corresponds to the number 1.52767; or,

$$x = 1.52767,$$

as far as five places of decimals.

As an example of the use of logarithms in theoretical physics, we may consider the following equation:

$$h = (1 + 0.00366t) \times 1,839,300 \log \frac{p_1}{p_2} \text{ centimetres.}$$

where h, is the vertical difference of elevation in centimetres between two stations at which the barometric pressures of p_1 and p_2, are respectively observed, t, being the temperature of the air in °C.

Thus, if

$$p_2 = 30.250'' \text{ and } p_1 = 30'', t = 15^\circ \text{C.},$$

then

$$\begin{aligned} h &= (1 + 0.00366 \times 15) \times 1{,}839{,}300 \log \frac{30.25}{30} \\ &= (1 + 0.549) \times 1{,}839{,}300 \times \log 1.008333 \\ &= 1.0549 \times 1{,}839{,}300 \times \log 1.008333 \\ &= 1.0549 \times 1{,}839{,}300 \times 0.0036040. \end{aligned}$$

This triple product, now cleared of logarithms, may be computed either arithmetically, or by logarithms; the result will be found to be

$$h = 6992.7 \text{ centimetres.}$$

All the logarithms we have hitherto considered have been common logarithms with

10, for the base. In mathematical theory, however, and frequently in applied mathematics, another and a more natural base suggests itself. This base, as far as seven decimal places, is 2.7182818, and is often represented by the symbol ε. Logarithms to this base are called *natural logarithms*, *Naperian logarithms*, or *hyperbolic logarithms*. For all practical purposes it is sufficient to remember that a natural logarithm of a number is always greater than the common logarithm of the same number in a definite ratio, which is 2.3026, approximately, or

$$2.71828^{n \times 2.3026} = 10^{n}.$$

If, therefore, n, is a logarithm in the common system, $2.3026n$ will be the approximate logarithm in the Naperian system. Thus the logarithm of 10 to the base ε, or

$\log_\varepsilon 10 = 1 \times 2.3026 = 2.3026$, approximately.

As an example of the application of hyperbolic logarithms in applied mathematics, we may consider the following equation:

$$W = p_1 v_1 \log_\varepsilon \left(\frac{v_2}{v_1}\right).$$

Here W, is the work done by a volume of gas expanding at constant temperature from a volume of v_1, to a volume v_2. Suppose, for example, that a volume of 5,000 cubic centimetres of a gas under a pressure of 2,000 grammes per square centimetre, expands to a volume of 10,000 cubic centimetres. Then the work done in the process will be

$$W = 2{,}000 \times 5{,}000 \times \log_\varepsilon \frac{10{,}000}{5{,}000}$$

$$
\begin{aligned}
&= 10^7 \log_\varepsilon 2 \\
&= 10^7 \times 2.3026 \log_{10} 2 \\
&= 2.3026 \times 10^7 \log_{10} 2 \\
&= 2.3026 \times 10^7 \times 0.3010300 \\
&= 6.93 \times 10^6 \text{ centimetre-grammes.}
\end{aligned}
$$

CHAPTER IX.

TRIGONOMETRY.

Trigonometry is the science relating to the measurement of angles in a plane, with particular reference to triangles, and also to figures bounded by straight lines. An angle is measured by the amount of opening between two straight lines meeting at a point. In all practical applications, when this opening completes one revolution, it is divided into 360 equal parts called *degrees*, which in their turn are each divided into 60 equal parts called *minutes*, and each of these into 60 equal parts called *seconds*. Consequently, a complete revolution is divided into 1,296,000 seconds, or 21,600 minutes, or 360 degrees. Almost all trigonomet-

rical tables in practical use refer to these angular units. For theoretical purposes it is found useful to employ a different unit called a *radian*. Thus in Fig. 2, suppose a fixed line OA, and a movable line OB,

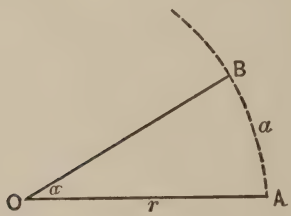

FIG. 2.

capable of rotating about the fixed point O, in the plane AOB. Then, if the point B, traces out an arc AB, the length of the arc AB, is the measure of the angle α, included between OA and OB. If the angle α, be such that the arc AB, is equal in length to the moving radius, or *radius vector* OB, then the angle α, is 1 *radian*.

This angle expressed in degrees is $57° 17' 44.8''$, approximately. If the length of the radius vector is taken as unity, then the magnitude of the angle α, will be the length of the arc AB. It will be evident that if we trace out a complete revolution or circle with the radius vector, the length of this circumference will be 2π units, and, therefore, the angle corresponding to a complete revolution is 360°, or 2π radians. Similarly, half a revolution is 180°, or π radians; a quarter of a revolution is a right angle of 90°, or $\frac{\pi}{2}$ radians; and, generally, an angle of n degrees is an angle of $\frac{\pi}{180} \times n$ radians.

An angle may extend beyond 360°. Thus two complete revolutions represent 720°, or 4π radians, and 3 complete revolutions 1,080°, or 6π radians, and so on. In

most practical applications, however, angles of less than 360° are considered, and, in the majority of cases, angles less than 90°.

An angle is considered *positive* when the direction of rotation of the radius vector is *counter-clockwise;* that is, the opposite direction to the rotation of the hands of a clock, when viewed from in front of the clock. If the rotation be made in the opposite direction, or *clockwise*, the angle is *negative.* Thus the angle AOB, Fig. 3, may be regarded either as $-45°$, or $-\frac{\pi}{4}$ radians; or, as $+315°$, or $+\frac{7\pi}{4}$ radians. When the angle considered has the radius vector between A and A_1, Fig. 3, it is said to lie in the *first quadrant.* When the radius vector lies between A_1 and A_2, it is said to lie in the *second quadrant.* When

between A_2 and A_3, in the *third quadrant*, and when between A_3 and A, in the *fourth quadrant.*

Besides the magnitude of an angle itself,

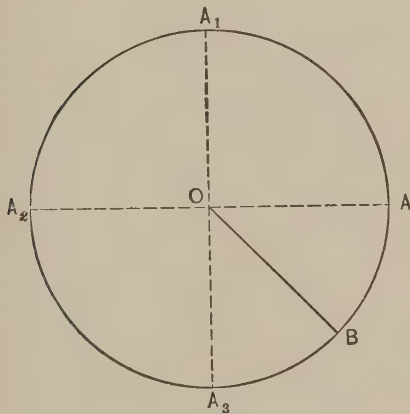

FIG. 3.

as determined in degrees or in radians, there are several important magnitudes connected with it which are called the *trigonometrical functions*, and which must be memorized since they constantly occur

in all trigonometrical writings. These trigonometrical functions or ratios, some times called the *circular functions,* are as follows:

The *sine;* the *cosine.*
The *tangent;* the *cotangent.*
The *secant,* and the *cosecant.*

Occasionally the *versed sine* and *coversed sine* are added, but they are rarely used.

In Fig. 4, the angle α, is represented as being contained between the fixed line OA, called the *initial line,* and the radius vector OB. If we let fall a perpendicular BC, from the point B, on to the initial line we obtain a right-angled triangle OBC. Then the ratio of the length of the perpendicular BC, to the length of the hypothenuse, or radius vector OB, or the fraction $\frac{BC}{OB}$, is called the *sine* of the angle α,

and is written in abbreviation sin α. Consequently, $\sin\alpha = \frac{BC}{OB} = \frac{\text{perpendicular}}{\text{hypothenuse}}$.

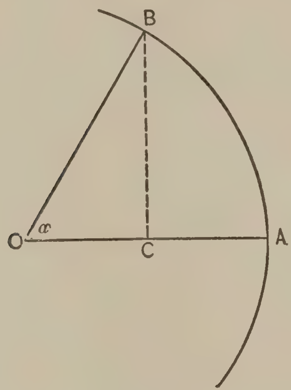

FIG. 4.

If the hypothenuse or radius vector be chosen of unit length, then the length of the perpendicular is the sine of the angle α, or $\sin\alpha = BC$. Thus, if $\alpha = 60°$ and OB, is unity, it will be found that the length of BC, is approximately, 0.866, so that $\sin 60° = 0.866$, as far as three decimal places. By reference to trigonometrical

tables it will be found that sin 60° = 0.8660254, as far as seven decimal places.

The ratio of the length of the base OC, to the hypothenuse or radius vector OB, is called the *cosine* of the angle α, so that $\frac{OC}{OB}$ = cosine α, which is abbreviated to cos α. If OB, be unity this ratio becomes simply OC, and $OC = \cos \alpha$. For an angle of 60° as shown, the length of OC, will be found to be exactly half that of OB, or

$$\cos 60° = \frac{1}{2} = 0.50.$$

The ratio of the length of the perpendicular BC, to the length of the base OC, or the fraction $\frac{BC}{OC}$, is called the *tangent* of the angle α, and is abbreviated tan α. Thus, for an angle of 60° the ratio $\frac{BC}{OC}$, will be found to be 1.732, approximately, and by reference to trigonometrical tables

the tangent of 60° is given as 1.7320508, as far as seven places of decimals.

Similarly, the ratio of the length of the base OC, to the length of the perpendicular BC, or the fraction $\frac{OC}{BC}$, (which is the reciprocal of the fraction representing the tangent) is called the *cotangent* of the angle α, and is abbreviated cot α. Consequently, cot $\alpha = \frac{OC}{BC}$, the value of the cotangent of an angle, will evidently be the reciprocal of the tangent, so that, since we have seen that the tangent of 60°, or tan 60° = 1.732, we know that the cotangent of the same angle, or cot 60° $= \frac{1}{1.732} =$ 0.5773503, approximately, as far as seven places of decimals.

The ratio of the length of the radius vector, or hypothenuse BO, to the length of

the base OC, or the fraction represented by $\frac{OB}{OC}$, is called the *secant* of the angle α, and is abbreviated sec α. Consequently, $\sec \alpha = \frac{BC}{OC}$. With OB, unity, this may be written $\sec \alpha = \frac{1}{OC}$. This ratio is evidently the reciprocal of the cosine of the same angle, or $\sec \alpha = \frac{1}{\cos a}$, and since we have seen that the cosine of $60° =$ 0.5, the secant of $60°$, or $\sec 60° = \frac{1}{0.5}$ $= 2.0$.

The ratio of the length of the radius vector or hypothenuse OB, to the length of the perpendicular BC, or the fraction represented by $\frac{OB}{BC}$ is called the *cosecant* of the angle α, and is abbreviated cosec α. Consequently, $\operatorname{cosec} \alpha = \frac{OB}{BC}$. With

OB, unity, this may be written cosec $\alpha = \frac{1}{BC}$. This ratio is evidently the reciprocal of the sine of the same angle or cosec $\alpha = \frac{1}{\sin \alpha}$ and, since we have seen that the sine of 60° is, approximately, 0.866, the cosecant of 60°, or cosec 60° $= \frac{1}{0.866} = 1.1547005$, as far as seven decimal places.

It is important to remember that

$$\cot \alpha = \frac{1}{\tan \alpha}$$

$$\sec \alpha = \frac{1}{\cos \alpha}$$

$$\operatorname{cosec} \alpha = \frac{1}{\sin \alpha}.$$

If we study the increase of an angle in the first quadrant, it will be seen that the length of the perpendicular BC, which

with unit radius is the sine, increases steadily, although not proportionately, from 0, when the angle is zero, to unity at 90°, when BC, coincides with the radius vector. The sine of an angle may have any numerical value between 0 and 1, but cannot exceed unity. The same is true as regards the cosine, but in the case of the tangent its value commences at zero, when the angle is zero, but increases indefinitely as the angle approaches 90°, because, while the perpendicular BC then approaches unity, the base OC becomes indefinitely small and, therefore, the ratio of the perpendicular to the base becomes indefinitely great. This is expressed by saying that tan 90° = infinity; or, as it is usually written, tan 90 = ∞.

Similarly, the cotangent of an angle commences with an indefinitely great value and diminishes as the angle is in-

creased, until it becomes 0 when $\alpha = 90°$. The cosecant commences at infinity when $\alpha = 0$, and diminishes to unity when $\alpha = 90°$. The secant commences at unity when $\alpha = 0$ and increases to infinity when $\alpha = 90°$.

As we increase the angle beyond the first quadrant as represented in Fig. 4, we cause the values of the trigonometrical functions to repeat themselves cyclically. Thus, in Fig. 5, the length of the perpendicular let fall from the end of the radius vector of unit length upon the initial line is always the sine of the corresponding angle. Thus, when OB, reaches 60° at B_1, $B_1 C_1 = 0.866$, approximately, or $\sin 60° = 0.866$. When, for example, OB reaches 150° at B_2, the length $B_2 C_2$, is 0.5, and $\sin 150° = 0.5$. When OB, passes into the third quadrant, the perpendicular CB, is below the initial line, or is negative in

value; consequently, at such a point, for example as OB_3, corresponding to $\alpha = 240°$, $B_3\,C_3 = -0.866$ and $\sin 240° = -0.866$.

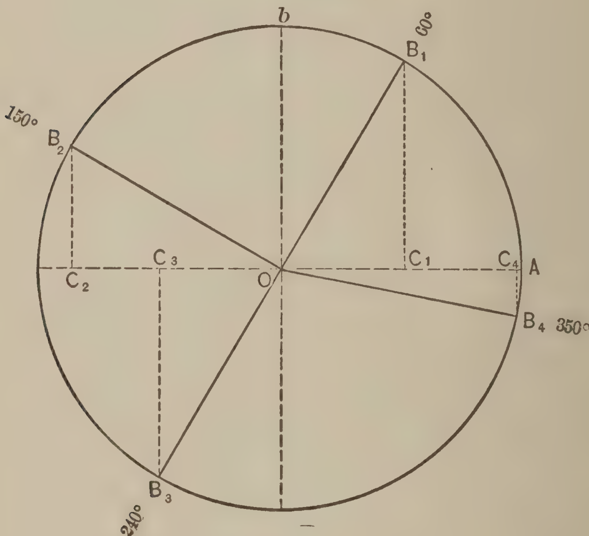

FIG. 5.

Again, when the angle passes through the fourth quadrant the perpendicular BC, is still below the initial line, and is, there-

fore, still negative, so that at the position OB_4, corresponding to $\alpha = 350°$, or $-10°$, the length $B_4 C_4$, is 0.1736482, approximately, and $\sin 350° = \sin -10° = -0.1736482$ as far as seven decimal places. The sine of the angle is, therefore, positive in the first two quadrants, and is negative in the third and fourth quadrants.

The cosine of an angle, when described with unit radius vector, is equal to the length of the base OC. In the first quadrant this base is always positive. Thus, in Fig. 5, $\cos 60° = OC_1 = 0.5$. In the second quadrant the base OC_2, lies on the left-hand side of the origin O, and is, therefore, reckoned as negative. Consequently, the cosine of all angles in the second quadrant is negative. At the position indicated, OB_2, the cosine OC_2, is, approximately, -0.866, and $\cos 150° = -0.866$, approximately. Similarly, in the

third quadrant, the base OC_3 still lies to the left-hand side of the origin, and is negative, so that the cosine of the angle 240° is $OC_3 = -0.5$. In the fourth quadrant, the base falls on the right-hand side of the origin and is positive. Consequently, $\cos 350° = \cos -10° = OC_4 = 0.9848079$, approximately. The cosines of angles in the first and fourth quadrants are, therefore, positive, and in the second and third quadrants, negative.

In the same way the tangent, cotangent, secant and cosecant could be followed out through the different quadrants, always remembering that a base is negative when drawn on the left-hand side of the origin, and a perpendicular is negative when drawn below the origin, while the radius vector remains positive throughout the revolution. It is sufficient to observe, however, that the tangent of the angle is

the ratio of the sine to the cosine. Thus, in Fig. 5,

$$\sin\ \alpha = \frac{BC}{OB}$$

$$\cos\ \alpha = \frac{OC}{OB}$$

$$\text{and } \tan\ \alpha = \frac{BC}{OC} = \frac{BC}{OB} \times \frac{OB}{OC}$$

$$= \frac{BC}{OB} \times \frac{1}{\frac{OC}{OB}}$$

$$= \sin\ \alpha \times \frac{1}{\cos\ \alpha} = \frac{\sin\ \alpha}{\cos\ \alpha}.$$

Consequently, if the sine and cosine of an angle are found, we obtain the tangent by dividing the latter into the former, while the cosecant, secant and cotangent are the reciprocals of the sine, cosine and tangent, respectively. Thus, we have seen that $\sin\ 240° = -0.866$, approximately, and

that the cos $240° = -0.5$. Therefore,

$$\tan 240° = \frac{-0.866}{-0.5} = +1.732.$$

Then $\operatorname{cosec} 240° = \dfrac{1}{-0.866} = -1.1547$

$$\sec 240° = \frac{1}{-0.5} = 2$$

$$\cot 240° = \frac{1}{+1.732} = +0.57735.$$

The foregoing rules will enable the student to understand how the trigonometrical ratios or functions are defined or measured for all angles. The science of trigonometry deals largely with the applications of these ratios. There are a number of important relations connecting the trigonometrical ratios. Thus, whatever the angle α may be, the following equation holds true,

$$\sin^2 \alpha + \cos^2 \alpha = 1;$$

or the square of the sine of the angle α,

added to the square of the cosine of that angle, gives unity.

Thus, in the case of Fig. 3,

$$\sin 60° = 0.866 \text{ and } \cos 60° = 0.5.$$

$$\begin{aligned} \text{Therefore, } \sin^2 60° &= (0.866)^2 = 0.750 \\ \text{and } \cos^2 50° &= (0.5)^2 = \underline{0.250.} \\ \text{Therefore } \sin^2 60° + \cos^2 60° &= 1.000. \end{aligned}$$

From these relations it will be evident that if we know the sine of an angle we can obtain its cosine, and from these the tangent and the reciprocal values, the secant, the cosecant and the cotangent.

It is not the intention of this book to develop the theory of trigonometry, but merely to enable a student intelligently to comprehend or interpret the meaning of any trigonometrical formula. We must refer him to treatises on trigonometry for such developments as will enable him to solve problems in trigonometry. We take

the following cases, however, as examples of trigonometrical equations:

Suppose that a tall object, such as a flag-staff AB, Fig. 6, has an unknown height which it is desired to measure, and that the observer at O, on the same level as the bottom A of the staff, observes the magni-

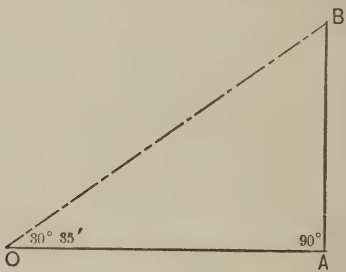

FIG. 6.

tude of the angle $AOB = \alpha$, which the staff subtends at his eye. Then, if the observer measures the length of the base OA, which separates him from the staff, he is at once enabled to compute the height of the staff AB, because he knows that $\dfrac{AB}{OA} = \tan\alpha$.

If we multiply both sides of this equation by OA, we have

$$\frac{AB}{OA} \times OA = OA \tan \alpha$$

or $$AB = OA \tan \alpha.$$

Consequently, the unknown height AB, is equal to the product of the measured base OA, and the tangent of the observed angle. Thus, if OA, were 100 feet and the angle α, was observed to be 30° 35′, then

$$AB = 100 \times \tan 30° \, 35';$$
$$= 100 \times 0.5910 = 59.1 \text{ feet},$$

approximately.

The attraction of gravitation is known to be different at different points of the earth's surface. Its value depends upon the latitude, and upon the elevation of the place at which the observation is made. The formula is usually expressed as follows:

$$g = 930.6056 - 2.5028 \cos 2l - 0.000003\,h, \text{ dynes};$$

where g, is the gravitational force in dynes or C. G. S. (centimetre-gramme-second) units of force, l, is the latitude of the locality, and h, is its height in centimetres above the level of the sea.

Thus, at the latitude of New York, which is, approximately, 40° 43′, and at an elevation of 10 metres, or 1000 centimetres above the mean tide level, the gravitational force would be:

$$
\begin{aligned}
g &= 980.6056 - 2.5028 \cos 2\,(40^\circ\ 43') - 0.000003 \times 1{,}000; \\
&= 980.6056 - 2.5028 \cos\,(81^\circ\ 26') - 0.003; \\
&= 980.6056 - 2.5028 \times 0.14896 - 0.003; \\
&= 980.6056 - 0.3728 - 0.003; \\
&= 980.2298 \text{ dynes.}
\end{aligned}
$$

As another example of a trigonometrical equation we may consider the following:

$$l = L \cos \alpha.$$

Here L, represents the length of a degree of longitude on the earth's surface measured at the equator, α is the latitude of any place on the earth's surface and l the length of a degree of longitude at that place. The equation states that the length of the degree varies as the cosine of the latitude, and since the cosine of an angle diminishes from 1 at 0°, to 0 at 90°, it is evident that at the pole the length of the degree of longitude would be indefinitely small, while at the equator it is, approximately, 69.6 geographical miles. The length of a degree longitude at New York, or any place having the latitude of New York, is

$$\begin{aligned} l &= 69.6 \times \cos 40^\circ \; 43'; \\ &= 69.6 \times 0.75794; \\ &= 52.81 \text{ miles.} \end{aligned}$$

In the problem of determining the apparent time at sea, and, therefore, with the aid of

the chronometer, the longitude of a ship, the following equation is employed :

$$\cos\left(\frac{H}{2}\right) = \sqrt{\sin S \sin (S - Z) \operatorname{cosec} P \operatorname{cosec} C.}$$

Here Z, is the angular distance of the sun's centre from the zenith, or point immediately overhead at the ship, as determined by the sextant ; P, is the co-declination, or polar distance, of the sun ; *i. e.*, the angular distance of the sun's centre from the pole of the earth, as it would be seen by an observer at the centre of the earth, if the earth's mass were transparent ; C, is the co-latitude, or the latitude of the ship subtracted from 90° ; *i. e.*, the angular distance of the ship from the earth's pole. S, is the half sum of P, C and Z;

$$\text{or } S = \frac{P + C + Z}{2}$$

or an angle equal to half the sum of the three angles above defined ; and H, is the

hour angle, or the apparent time at the ship, expressed in degrees. The equation states that the cosine of half the hour angle is equal to the square root of the product of four quantities; the first is the sine of the half sum; the second is the sine of the half sum reduced by the angle Z; the third is the cosecant of the polar distance; and the fourth is the cosecant of the co-latitude. These quantities may be found from trigonometrical tables when the three angles P, C and Z, are known. It is usual, however, to make the computation with the aid of logarithms as already described, since the multiplication of the four trigonometrical functions is performed logarithmetically by adding four logarithms, and the square root is then readily obtained by dividing the logarithmic sum by two. For this reason it is customary to seek in the logarithmic tables, not the

simple sine or cosecant, but the logarithm of that sine or cosecant. In practice this operation is carried out by the mere addition of logarithms by the navigator, without the necessity of writing down the above equation.

Thus, suppose a ship, at $52° 12' 42''$ north latitude, observes, with a sextant, that the altitude of the sun's centre above the horizon is $39° 5' 28''$; the declination of the sun, or its angular distance above the earth's equator, as it might be observed from the centre of the earth, obtained by tables, being $15° 8' 10''$ north. Required the apparent time at the ship the moment the observation was made.

Here Z, the zenith distance of the sun's centre, is

$$90° - (39° 5' 28'') = 50° 54' 32'';$$

P, the polar distance of the sun's centre, is

$$90° - (15° 8' 10'') = 74° 51' 50'';$$

C, the co-latitude of the ship,

is

$$90° - (52° \, 12' \, 42'') = \underline{37° \, 47' \, 18''.}$$

The sum of these angles is $163° \, 33' \, 40''$

and S, the half sum, is $81° \, 46' \, 50''$.

Since $S = 81° \, 46' \, 50''$, sin $S = 0.989726$.

" $S - Z = 81° \, 46' \, 50'' - (50° \, 54' \, 32'') =$ $30° \, 52' \, 18''$, sin $(S - Z) = 0.51312$.

" $P = 74° \, 51' \, 50''$, cosec $P = 1.036$.

" $C = 37° \, 47' \, 18''$, cosec $C = 1.632$.

These values are found by reference to trigonometrical tables.

The equation becomes, therefore,

$$\cos\left(\frac{H}{2}\right) = \sqrt{0.989726 \times 0.51312 \times 1.036 \times 1.632.}$$

$$= 0.9266$$

$$= \cos 22° \, 5', \text{ approximately.}$$

$$\therefore \frac{H}{2} = 22° \, 5',$$

or $\quad H = 44° \, 10'$.

Every 15° represents one hour of time, and every 15′ represents 4 minutes of time.

Consequently, H = 2 hours, 56 minutes, 40 seconds, approximately.

Arithmetical computation in this case is seen to be lengthy and tedious, but it is greatly facilitated by employing logarithms as follows:

Since $S = 81° \ 46' \ 50''$ the logarithm of the sine of S, by tables, is	=	9.99552
$(S - Z) = 30° \ 52' \ 18''$, log sin $(S - Z)$ or L sin $(S - Z)$	=	9.71020
$P = 74° \ 51' \ 50''$, log cosec P, or L cosec P	=	10.01535
$C = 37° \ 47' \ 18''$, log cosec C, or L cosec C	=	10.21273
The logarithm of the product		39.93380.
Dividing this sum by 2, to extract the square root,	=	19.96690.

This corresponds to the logarithm of the cosine of $22° \ 5'$

$\therefore \ H = 22° \ 5' \ 00''$ or $H = 44° \ 10' \ 00''$, and this angle expressed in time, allowing $15°$ to the hour, $15'$ to the minute, and $15''$ to the second, gives 2h. 56m. 40s.

It will be observed that in trigonometrical tables, 10 is always added to the characteristic of a logarithm, so as to avoid the use of negative characteristics. The logarithm 10.01535, therefore, really represents 0.01535, while 9.01535 represents $\bar{1}.01535$, and so on.

The area of a triangle two of whose sides have lengths a and c, while their included angle is B, is expressed by the equation

$$A = \frac{1}{2} ac \sin B.$$

This equation states that the area is equal to half the product of the two sides

into the sine of the included angle. Thus, if $a = 50$ feet, $c = 60$ feet and B, the included angle $= 45°$, $\sin B = 0.7071068$, and

$$\begin{aligned} A &= \tfrac{1}{2} \times 50 \times 60 \times 0.7071068; \\ &= 1{,}500 \times 0.7071068; \\ &= 1{,}060.66 \text{ square feet.} \end{aligned}$$

It is sometimes convenient to employ in trigonometrical equations what is called *inverse notation*, with the meaning of which, therefore, the student should be familiar.

The ordinary expressions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, $\operatorname{cosec} \alpha$, $\cot \alpha$, refer to the trigonometrical ratios of the angle α. The inverse expressions $\sin^{-1} \alpha$, $\cos^{-1} \alpha$, $\tan^{-1} \alpha$, $\sec^{-1} \alpha$, $\operatorname{cosec}^{-1} \alpha$, and $\cot^{-1} \alpha$ express the angles of which the sine, cosine, tangent, secant, cosecant, and cotangent are respectively equal to the quantity α.

Thus $\tan^{-1} 1$ is an angle whose tangent is 1, or is 45°.

$\cos^{-1} 1$ is an angle whose cosine is 1, or 0°.

$\sin^{-1} (\frac{1}{2})$ is an angle whose sine is $\frac{1}{2}$ or 30°.

The equation

$$\tan^{-1} 2 = \alpha$$

means that α, is an angle whose tangent is 2,or 63° 26′. This will be evident on taking the tangent of both sides of the equation:

$$\tan (\tan^{-1} 2) = \tan \alpha.$$

Here the two operations tan, and $\tan^{-1}$, are inverse to each other or cancel, leaving

$$2 = \tan \alpha$$

from which, by trigonometrical tables, $\alpha = 63° \ 26'$, approximately.

CHAPTER X.

HYPERBOLIC TRIGONOMETRICAL FUNCTIONS.

We have seen that plane trigonometry deals with the *trigonometrical* or *circular functions: i. e.*, those in which the radius vector describes a circle, as shown in Figs. 2, 3, 4 and 5. There are, however, certain mathematical applications in which the functions are *hyperbolic; i. e.*, those in which the radius vector describes a hyperbola. In Fig. 7, the curve ABC, is part of a *rectangular hyperbola; i. e.*, part of a section of a right cone whose vertical angle is 90°, when cut by a plane parallel to the axis of the cone. If the distance OA, which is called the *semi-axis* of the curve, is taken as unity, and the radius vector OB,

turning around the fixed point O, moves over the hyperbola ABC; then twice the area comprised between OA, OB, and the curve ABC, is the *hyperbolic angle* de-

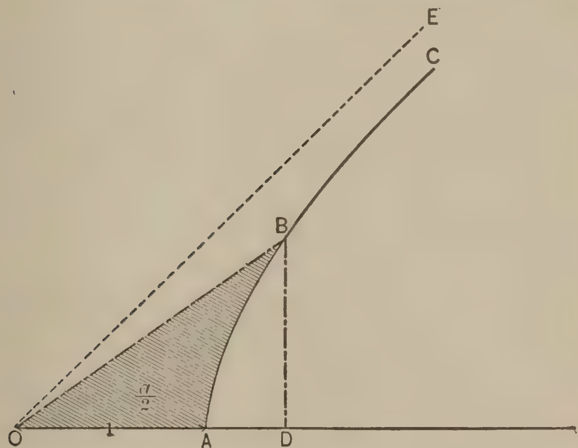

FIG. 7.

scribed by the radius vector. This hyperbolic angle is not equal to the angle AOB, between the semi-axis OA and the radius vector OB. Thus, the shaded area marked OAB, in the figure is, approxi-

mately, 0.44 square inches, if the base or semi-axis OA, is one inch. Consequently, the hyperbolic angle traced out in this case by the radius vector OB, is $2 \times 0.44 = 0.88$, approximately. As the radius vector advances along the hyperbola ABC, the shaded area rapidly increases, and with it the hyperbolic angle, but even when the radius vector is infinitely long, or extends over an infinite length of the curve ABC, the circular angle between OA and OB, cannot exceed the angle AOE, since OE, is what is called the *asymptote* of the curve, or the line which continually approaches the curve ABC, but only meets it at infinity.

If we let fall a perpendicular BD, from the end B, of the radius vector upon the base line OD, the length BD, is called the *hyperbolic sine* of the hyperbolic angle. Thus, in the case represented in Fig. 7,

$BD = 1$, if $OA = 1$. Consequently, the hyperbolic sine of the hyperbolic angle 0.88 is 1. This is represented symbolically by the equation:

$$\sinh 0.88 = 1,$$

which may be read "the h-sine of 0.88 is unity."

Similarly, the length OD, from the origin to the foot of the perpendicular BD, is the hyperbolic cosine of the hyperbolic angle; or

$$OD = \cosh \alpha,$$

which is read the length OD (when $OA = 1$) is the h-cosine of the hyperbolic angle α. In the case represented in Fig. 7, $\alpha = 0.88$, approximately, and $OD = 1.41$, approximately, Consequently,

$$1.41 = \cosh 0.88.$$

From these two quantities all the remain-

ing hyperbolic functions may be immediately deduced.

Thus,
$$\tanh \alpha = \frac{\sinh \alpha}{\cosh \alpha};$$
$$\operatorname{sech} \alpha = \frac{1}{\cosh \alpha};$$
$$\operatorname{cosech} \alpha = \frac{1}{\sinh \alpha};$$
$$\coth \alpha = \frac{1}{\tanh \alpha}.$$

These relations are precisely similar to those which apply in the case of the circular functions, that is to say, the tangent is the ratio of the sine to the cosine, while the secant, cosecant and cotangent are respectively reciprocals of the cosine, sine and tangent. In the case of Fig. 7,

$$\tanh 0.88 = \frac{1}{1.414} = 0.707, \text{ approximately};$$
$$\operatorname{sech} 0.88 = \frac{1}{1.414} = 0.707, \text{ approximately};$$

$$\operatorname{cosech} 0.88 = \frac{1}{1} = 1.0, \text{ approximately};$$

$$\coth 0.88 = \frac{1}{0.707} = 1.414, \text{ approximately}.$$

As the radius vector advances, the hyperbolic sine and cosine both increase indefinitely, while their ratio tanh α, approximates more and more nearly to unity.

Tables are published of the numerical values of the hyperbolic functions, and by means of these tables it will always be possible to apply numerically such formulæ as appear in technical works.

As an example of the use of hyperbolic functions we may consider the following formula, which gives the deflection at the end of a horizontal beam of uniform cross-section built into a solid wall or support at one end, and having a weight or load

P, attached to the free end, accompanied by a horizontal tension Q.

$$D = \frac{P}{Q}\left(l - \frac{\tanh nl}{n}\right)$$

where l, is the length of the beam,

$$n = \sqrt{\frac{Q}{EI}},$$

where E, is a co-efficient of elasticity, I, the moment of inertia of the section. Then the formula states that the deflection at the free end of the beam is equal to the quotient of P by Q, into a compound term within the bracket, the first term of which is the length of the beam, and the second, which is negative, is one nth of the hyperbolic tangent of the product of the length and n. If, for example, $nl = 2.0$ say, then a table of hyperbolic tangents would show that $\tanh nl = \tanh 2 = 0.964$, approximately.

CHAPTER XI.

DIFFERENTIAL CALCULUS.

SUPPOSE an observer is carried on a railway train and that he wishes to determine by observations the speed at which the train is moving from time to time. It is, of course, possible that he might connect a centrifugal indicator with the car axle and cause the instrument to automatically indicate or record the speed from moment to moment, and such instruments are actually used for the purpose, but without such an instrument he would be obliged to make an observation of the distance through which the train ran in a given time taken from a stop-watch. Thus, if in one minute by the watch the

train ran exactly three-quarters of a mile, then the speed of the train is evidently 3/4ths of a mile per minute, or 45 miles per hour. This is not stating that the train actually maintained 45 miles per hour throughout the minute under observation, because the speed might actually have been, say 50 miles an hour during some portion of the minute, and, perhaps, 40 miles an hour at some other portion. It is only a statement that the mean speed during the minute was 45 miles per hour. If, however, the observer could measure 10 seconds of time accurately, by a stop-watch, and observed that the train ran 650 feet in this time, then the speed of the train during 10 seconds would be an average of 65 feet per second; or, $65 \times 3600 = 225{,}000$ feet per hour $= \frac{225{,}000}{5280}$ $= 42.61$ miles per hour. In this case the

computed speed would more nearly represent the actual speed of the train at the time of observation, because during 10 seconds there would be less time for the speed to vary than during 60 seconds, but still the observer could not say positively that the speed did not vary during 10 seconds, and he would be still somewhat uncertain of the exact speed when he commenced the observation. Now supposing that it was possible for him to make a measurement during a single second, however difficult this might be in practice, then if the distance run through by the train was exactly 66 feet in this second, the speed would be 66 feet per second, or 45 miles per hour. The observer would now be satisfied that he had determined the speed of the train at the moment when the observation was made with a greater degree of accuracy, because there was so

little time for variation to take place. But if the train had its brakes applied at the instant when the second of time was observed, it is quite possible that its speed would be rapidly falling, and that the speed at the commencement of the second would be appreciably greater than at the end of the second, so that even here, in the case of a rapidly retarded train, the time of observation could be too great. Reasoning, however, in this way we might assume that supposing it were possible to make an observation of the distance run through by the train in an exceedingly small interval of time which we may represent by Δt, (pronounced Delta *t*), say the 1/1000th part of a second, the corresponding short distance which we may represent by Δs, (pronounced Delta *s*) would enable the velocity of the train to be computed from the quotient $\frac{\Delta s}{\Delta t}$, with a very high degree

of accuracy; and that, theoretically, if Δt, were made indefinitely small, as represented symbolically by dt, the corresponding vanishingly small distance ds, would enable the true velocity to be computed from the equation

$$v = \frac{ds}{dt}.$$

In this equation ds, taken by itself, has no meaning, because it is the space run through by the train in an indefinitely small interval of time; and so too, dt, taken by itself, has no meaning, because it stands for an indefinitely small interval of time, but the fraction $\frac{ds}{dt}$, can be considered as the limit of $\frac{\Delta s}{\Delta t}$, as t, is reduced indefinitely, standing for a perfectly definite notion; namely, the instantaneous velocity of the train when the time chosen

for observing it is so extremely short that there is no possibility of error due to variation of speed in that time.

In the language of the differential calculus ds, is the *differential* of the space run through by the train; dt, is the *differential* of the time during which the observation of distance is measured, and $\frac{ds}{dt}$, is the *differential coefficient of the space* s, run through *with respect to the time* t. A differential coefficient has, therefore, always the meaning of a ratio between the variation of two quantities carried to a limit. Thus the differential coefficient $\frac{ds}{dt}$, is the limit of the ratio $\frac{\Delta s}{\Delta t}$, when Δt, is made indefinitely small.

The object of the differential calculus is to determine from a known relation between two connected quantities the rate

at which one varies when a variation is given to the other. Suppose, for example, that we consider the case of a stone falling to the ground from a given elevation. We know by observation, in connection with the theory of the subject, that the law which expresses the distance through which the stone will fall in a given time t seconds, neglecting air-friction, is

$$s = \frac{1}{2} g t^2 \qquad \text{feet,}$$

where g, is the constant representing the earth's gravitational acceleration, or 32.2 feet per-second-per-second, approximately. Consequently, the formula becomes

$$s = 16.1 t^2 \qquad \text{feet,}$$

approximately.

Having given this relation between the time t, and the distance fallen through s, the velocity at any instant is evidently

defined in some way by this relation. We might imagine that the observation was made at any particular instant, say $t = 3$, or three seconds after the stone commenced to fall. At this time $s = 16.1 \times 3^2$

$$= 16.1 \times 9 = 144.9 \text{ feet.}$$

By taking a small interval of time, say the one-tenth part of a second, thereby making $t = 3.1$, we find that the space fallen through up to this time is

$$s = 16.1 \times (3.1)^2 = 154.72 \text{ feet,}$$

so that, in the interval of time 0.1 second, the space fallen through will have been $154.72 - 144.9$

$$= 9.82 \text{ feet}$$

or $\Delta s = 9.82$ feet, and Δt, the interval of time, 0.1 second; so that the velocity,

as gauged from this interval, will be

$$\frac{\Delta s}{\Delta t} = \frac{9.82}{0.1} = 98.2 \text{ feet per second.}$$

But it will be evident that this is only a mean velocity during the time Δt, since the stone is constantly accelerated, and it will be somewhat in excess of the actual velocity of the stone at the moment $t = 3$.

The differential calculus supplies rules for determining the instantaneous velocity, or, in this case, the limit of $\frac{\Delta s}{\Delta t}$, when Δt, instead of being 0.1 second, is reduced indefinitely, and which, therefore, enable the instantaneous velocity $v = \frac{ds}{dt}$, to be determined accurately. It might be shown by rules of the differential calculus that the true velocity is gt, or 32.2 t, so that

$$\frac{ds}{dt} = 32.2\ t,$$

so that the velocity of the stone at the end of the third second, or when $t = 3$, is

$$\frac{ds}{dt} = 96.6 \text{ feet per second,}$$

and we should find that the result obtained above, of 98.2 feet-per-second gradually approached, the instantaneous velocity 96.6, as the interval of time Δt, was reduced from 0.1 second to a vanishingly small quantity.

In the case of the moving train, the differential calculus would be of no assistance to the observer, unless he knew the law which connects the distance traversed by the train with respect to time. We have only alluded to the case of a moving train in order to give a conception of the limiting rate which is so constantly dealt with in the theory of the differential calculus.

Whenever a differential coefficient such as $\frac{df}{ds}$, is met with in a formula, it is to be regarded as an actual quotient or fraction; namely, the limit which the fraction $\frac{\Delta f}{\Delta s}$ assumes when Δs, is indefinitely reduced.

It is always to be remembered that ds, in the differential calculus is not the product of the quantity d, into a quantity s, as it would be in ordinary algebra, but an abbreviation for "differential of s."

As another example, consider the case of a reservoir of water, discharging through a pipe, and let q, be the quantity of water in the reservoir, and t, the time. Then, during discharge, there will be a certain flow of water through the pipe. If q, be expressed in cubic centimetres and t, in seconds, we may ascertain that the rate of discharge in a given time Δt,

say 10 seconds, amounts to $\Delta q = 15{,}000$ cubic centimetres. Then the average flow of liquid through the pipe will be

$$\frac{\Delta q}{\Delta t} = \frac{15{,}000}{10} = 1{,}500$$

cubic centimetres per second. In order to obtain the true instantaneous value of the flow, it would be necessary to consider the interval of time Δt, in which the measurement was made, indefinitely reduced, as represented by the symbol dt, and the corresponding quantity of escape in that time dq, so that the instantaneous flow will be

$$f = \frac{dq}{dt}.$$

In practice the differential calculus could not here be employed, unless the relation between the quantity of water in the reservoir and the time was known, but the

conception afforded by the symbol $\frac{dq}{dt}$; namely, the instantaneous rate of change in the quantity with respect to time, or the instantaneous flow, would be perfectly definite, whether it could be employed practically or not.

Again, we may consider the quantity of heat q, which must be given to a gramme of a substance to raise its temperature θ°C. If we take the quotient $\frac{\Delta q}{\Delta\theta}$, this quotient is the mean thermal capacity of the substance. Thus, if we take $\Delta\theta$, as unity or as 1° C., then ΔQ, is the amount of heat which must be given to a gramme of the substance to raise its temperature 1° C. This quantity of heat is generally not the same at different temperatures, so that the quantity ΔQ, which would have to be given to a substance to raise it

from 5° to 6° C. is not the same as that required to raise it from say 31 to 32° C. Consequently, the thermal capacity varies with the temperature, and a strict definition is given by the equation

$$c = \frac{dq}{d\theta}, \text{ or the limiting value of } \frac{\Delta Q}{\Delta \theta},$$

when the variation $\Delta\theta$, is made indefinitely small. This may be regarded as the instantaneous rate of change of heat with respect to temperature, at any given temperature.

It is shown in works on electricity that if a loop of wire be subjected to variations in the quantity of magnetic flux Φ, which passes through it, either by being moved through magnetic flux, as by passing the poles of a magnet, or by having magnetic flux moved through it, as by moving a magnet past it, the electromotive force

(abbreviated E. M. F.), generated in the loop, or the electric pressure tending to cause an electric current to flow through it is represented by the formula

$$e = \frac{d\Phi}{dt},$$

where Φ, is the magnetic flux, e, the E. M. F. expressed in C. G. S. units and t, is the time expressed in seconds. This formula means that the E. M. F. is the instantaneous time-rate-of-change of magnetic flux through the loop. If the loop at a certain instant, $t = 10$, contains, say 1,000,000 C. G. S. units of magnetic flux, and if in the next half second or at $t = 10.5$, the magnetic flux had increased to 1,100,000 C. G. S. units, the increase would amount to 100,000 in the time 0.5 second, and the average rate of increase during the half second would be

$$\frac{100{,}000}{0.5} = \frac{\Delta\Phi}{\Delta t} = 200{,}000\,;$$

so that the average E. M. F. would be 200,000 C. G. S. units. If the change took place uniformly, this would be the actual E. M. F. throughout the half second, but if the change occurred irregularly, the E. M. F. might have a value at some part of the time greatly in excess of 200,000 units, and, at some other part of the time, greatly in deficit. But at any instant, if we conceived that the interval of time Δt, be reduced indefinitely, the ratio so obtained of $\frac{\Delta\Phi}{\Delta t}$, which would thereby become $\frac{d\Phi}{dt}$, is the true E. M. F. at that instant, no matter how rapidly the rate-of-change in flux may be varying. If there be any known law connecting the variable Φ, or flux, called the *dependent*

variable, with the variable t, or time, called the *independent variable*, then, from this law, it will be possible, by the rules of the differential calculus, to determine how $\frac{d\Phi}{dt}$, will vary from moment to moment, or what the instantaneous E. M. F. in the loop will be from moment to moment. Thus, if the flux through the loop increases steadily with time, so that say $\Phi = at$, then the rules of the differential calculus show what is evident on reflection, that since each second adds an amount of flux to the loop equal to a, the E. M. F. will be a units steadily; or

$$\frac{d\Phi}{dt} = a.$$

As another example, consider a simple pendulum consisting of a fine thread attached to a small heavy particle acting as the bob. This pendulum may have a cer-

tain length l cms., and it will have a certain periodic time, or time of one complete double swing, T seconds. Then, if we vary the length l, in any manner, and study the effect of this variation upon the periodic time, we make l, the independent variable or the variable whose change is arbitrary, and make T, the dependent variable, or the variable whose changes are not arbitrary, but are determined by the changes in l. The law connecting the period T, with the length l, is expressed algebraically as follows :

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ seconds,}$$

where π, is the ratio 3.14159 . . . of a circumference to its diameter, l, is the length of the pendulum in centimetres, and g, the accelerating force of gravity, at the location considered, in centimetres-per-second-

per-second; so that the periodic time is 6.2832 multiplied by the square root of the length divided by the gravitational constant. If now we make a change in l, we make a corresponding change in T, because as we lengthen the pendulum it swings more slowly, but the rate of change in T, with respect to l, or the rate-of-change of periodic time per centimetre increase in pendulum length, is

$$\frac{\Delta T}{\Delta l},$$

where Δl, is a given increase in l, and ΔT, the corresponding increase in T. This, however, is only an average rate of increase in periodic time, for the interval Δl, considered, and is not the true rate at that length. Thus, suppose $g = 981$ centimetres-per-second-per-second, and $l = 100$ centimetres or one metre.

Then,

$$T = 2 \times 3.1416 \times \sqrt{\frac{100}{981}};$$
$$= 6.283 \sqrt{0.1019368};$$
$$= 6.283 \times (0.3192);$$
$$= 2.0055 \text{ seconds.}$$

The half cycle, or single vibration, would, therefore, take 1.00275 seconds, so that the pendulum one metre long would be nearly a second's pendulum. If now we increase the length by one centimetre or make $\varDelta l = 1$,

$$T = 2 \times 3.1416 \times \sqrt{\frac{101}{981}};$$
$$= 6.283 \sqrt{0.10294};$$
$$= 6.283 (0.32085);$$
$$= 2.0159 \text{ seconds.}$$

Consequently,

$\varDelta T = 2.0159 - 2.0055 = 0.0104$ seconds,

and

$$\frac{\Delta T}{\Delta l} = \frac{00.104}{1.0} = 0.0104 \text{ second per centimetre.}$$

This is the average rate of change in the period of the pendulum per centimetre of increase in its length, but the actual rate of change when $l = 100$, is this ratio when Δl, is so far reduced that the pendulum does not sensibly alter in length; or,

$$\frac{dT}{dl}.$$

This differential coefficient of T, with respect to l, is known, by the rules of the differential calculus, to be,

$$\frac{dT}{dl} = \frac{\pi}{\sqrt{lg}}$$

When $l = 100$, and $g = 981$, this is

$$\frac{dT}{dl} = \frac{3.1416}{\sqrt{981 \times 100}};$$

$$= \frac{3.1416}{\sqrt{98,100}};$$

$$= \frac{3.1416}{3132};$$

$$= 0.01003 \text{ seconds-per-centimetre.}$$

Consequently, the rate of increase of periodic time per centimetre of length is 0.01003 seconds-per-centimetre, in a metre pendulum, whereas the approximate process of calculation, assuming the actual variation of one centimetre, gave 0.0104 seconds-per-centimetre.

If a body move in time t, seconds, through s, centimetres, the instantaneous velocity is expressed by the differential coefficient of the space described with respect to the time, or

$$v = \frac{ds}{dt} \text{ centimetres per second.}$$

But if we consider the body moving with a velocity v, centimetres per second, in time

t, seconds, this velocity may be increasing or diminishing, and its time-rate-of-increase is called acceleration. This instantaneous time-rate-of-increase is the limit of $\frac{\Delta v}{\Delta t}$, and becomes

$$a = \frac{dv}{dt}.$$

But if we substitute for v, in this equation its equal

$$\frac{ds}{dt},$$

we obtain

$$a = \frac{d\left(\frac{ds}{dt}\right)}{dt}$$

centimetres-per-second-per-second, and this is written

$$a = \frac{d^2s}{dt^2},$$

so that

$$\frac{d^2s}{dt^2}$$

is called the *second differential coefficient* of s, with respect to t, to distinguish it from

$$\frac{ds}{dt},$$

which is the first differential coefficient of s, with respect to t. In the same way

$$\frac{d^3s}{dt^3},$$

which is the *third differential coefficient* or the differential coefficient of the differential coefficient of the differential coefficient with respect to t, and so on. It would mean the instantaneous time rate of acceleration.

A differential equation is an equation which contains differentials or differential cofficients. We will consider such an equation with a view of interpreting its meaning.

In Helmholtz's "Sensation of Tone,"

App. IX., appears the following equation in relation to the vibration of a tuning fork:

$$m.\frac{d^2x}{dt^2} = - \ a^2x - b^2\frac{dx}{dt} + A \sin nt,$$

where x, is the excursion of a heavy vibrating point from its mean position of rest, a is a coefficient of elasticity, b, is a coefficient of frictional opposition to motion, A, is the amplitude or maximum value of an impressed vibratory force, t, the time in seconds starting from some particular epoch, n, a number measuring the frequency of the variations of the impressed force, and m, is the mass of the heavy body acted upon. The meaning of these symbols is given in the text referred to.

The quantity on the left-hand side of the equation is the product of the mass m, and

$$\frac{d^2x}{dt^2},$$

the second differential coefficient of the excursion x, with respect to time. The first differential coefficient,

$$\frac{dx}{dt}$$

would represent the instantaneous velocity of the body at the time considered. The second differential coefficient,

$$\frac{d^2x}{dt^2},$$

is the instantaneous rate-of-change of velocity, or the instantaneous acceleration. The right-hand side of the equation has three terms. The first is the product of the excursion x and the coefficient of elasticity $-a^2$. This is a force tending to restore the body to its initial condition of rest, and increases as the excursion increases. The second term is the product of the coefficient $-b^2$, and the instantaneous velocity

$$\frac{dx}{dt}.$$

This is a force of friction tending to oppose the motion, and increasing directly with the velocity. The third term is the impressed vibratory force, acting on the body, and is the product of the constant A, and the trigonometrical expression sin (nt). As t, increases, the angle represented by nt, increases proportionately, and passes successively at a definite rate through all four quadrants. The sine of this angle, represented by sin (nt) will, therefore, pass successively through 0, + 1, 0, − 1, 0, + 1, etc., and all intermediate values. Consequently, the third term will pass through the values 0 A, 0 − A, 0, + A, etc.; and all intermediate values, which is equivalent to the statement that the impressed force alternates at a definite rate between the values + A and − A, according to a sine

law. The equation, therefores, states that at any and every instant the force of inertia, which is the product of the mass and acceleration, is equal to the sum of all the forces acting on the body, the first being the elastic force of restitution, the second the frictional force of retardation, and the third the impressed force.

It will thus be evident that even when the student is not sufficiently acquainted with the rules of the differential calculus to deduce differential coefficients, or manipulate them in equations, he will, nevertheless, be able to interpret the meaning of the equations which appear in technical works employing differential calculus.

It is shown in treatises on dynamics that if a rigid body be free to move about a fixed axis and has a moment of inertia K, gramme-cm^{2}s; and is acted upon by any force which at time t seconds, exerts a mo-

ment about the axis of M dyne-cms, the following equation holds at any and every instant

$$\frac{d^2\theta}{dt^2} = \frac{M}{K}.$$

Here $\frac{d^2\theta}{dt^2}$, is the second differential coefficient of the angle θ, described by the body about the axis with respect to the time t, or is the instantaneous time-rate-of increase of the instantaneous time-rate-of-increase of angle. If we call the instantaneous time-rate-of-increase of angle the instantaneous angular velocity, and if we call the instantaneous time-rate-of-increase of angular velocity, the instantaneous angular acceleration, then the equation states that the angular acceleration at any instant is the quotient of the moment of the applied force divided by the moment of inertia of the body. It may be possible to determine by the rules of the

differential calculus and so make use of the equation numerically, but even if the information is not forthcoming by which the equation may be practically applied, or if the knowledge is not available by which from the data of the problem the computation may be made, still the interpretation of the meaning of this equation carries to the mind of the student a definite and perfectly intelligible law.

CHAPTER XII.

INTEGRAL CALCULUS.

We have seen that the principal object of the differential calculus is to determine the limiting ratio of the variation in a dependent variable, with respect to an indefinitely small variation of an independent variable with which it is connected. The principal object of the integral calculus is to effect the inverse operation; namely, to determine from the limiting ratio of variation between a dependent and an independent variable, the fundamental relation between these variables. Thus, from the known law that the distances fallen through by a stone in time t seconds, is expressed by the formula

$$s = \tfrac{1}{2}gt^2,$$

where t, is the independent variable, and s, the dependent variable, the differential calculus determines the limiting relation between their variations, or gives the differential coefficient

$$\frac{ds}{dt} = gt,$$

and states that the instantaneous velocity at any moment is equal to the product of the gravitational acceleration-constant g, and the time of descent t, or that for any very minute change in the independent variable amounting to dt, the corresponding change in the dependent variable is

$$ds = gt.dt.$$

The integral calculus would find its application in the inverse relation: Having given the observed fact that the velocity

of a falling stone is expressed by the formula,

$$v = gt \text{ or } \frac{ds}{dt} = gt,$$

what will be the relation between the primitive variables s and t, corresponding to this condition? We shall see that any such inquiry naturally leads to the summation of an indefinitely great number of indefinitely small terms.

Fig. 8, shows a straight line OV, drawn from the origin O, to a scale such that in each second of time, as measured along the base or axis of abscissæ OT, the elevation of the line represents the velocity of the falling stone. Thus, at $t = 1$ second, the elevation of the point on the line above 1 on the base, represents, to scale, 32.2 feet per second; at 2 seconds it is 64.4, and so on for all times, integral or fractional, included between $t = 0$ and $t = 2.5 = T$.

This line, therefore, expresses graphically the equation

$$v = gt,$$

where $g = 32.2$, as far as $t = 2.5$ seconds.

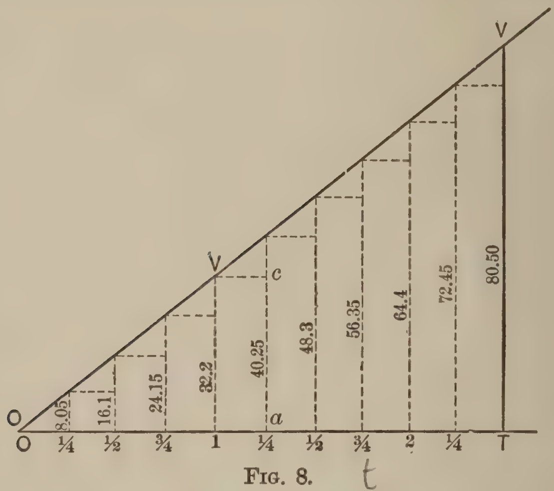

Fig. 8.

It will be evident that if we consider the moving stone at some particular instant, say at the time when $t = 1$, and $v = 32.2$, the space through which the stone will move at this velocity in a small interval of

time Δt, will be $\Delta s = v\Delta t$, so that, for example, if $\Delta t = \frac{1}{1000}$ second, the stone will move through $\Delta s = \frac{32.2}{1000}$ feet in this time. Strictly speaking, the stone will have moved through more than this distance, because during this interval of time it will have been accelerated or will be moving faster at the end than at the beginning of the period. But if we make the interval of time Δt, small enough, and theoretically, if we make it indefinitely small or dt, the equation,

$$ds = vdt,$$

will become strictly true. At present, however, we may accept for our purposes the rough calculation

$$\Delta s = v\Delta t$$

where Δt = 1/4th second say, or 0.25.

Then we may divide the whole base line OT of 2.5 seconds, into intervals of time Δt each equal to 0.25 second, and there will evidently be ten of such intervals. If we take the first interval, when $t = 0$, the stone will evidently be just starting from rest, and its initial velocity will be 0. This may be represented by the making $v_0 = 0$. Consequently,

$$\Delta s_0 = v_0 \Delta t = 0 \times 0.25 = 0 \text{ feet.}$$

And the space moved through by this rough calculation will be 0 feet in the first quarter second, although we know that owing to taking Δt so long as 1/4th second, the result is untrue since the stone has certainly started from rest in this time. At the commencement of the second interval, or when

$$t = 0.25 \quad v_1 = 0.25 \times 32.2 = 8.05 \text{ feet per second.}$$

At this velocity, if maintained uniform, the

stone would traverse in the succeeding interval $\varDelta t$, a distance

$$\varDelta s = 8.05 \times 0.25 = 2.0125 \text{ feet.}$$

If we proceed in this way to determine at each of the ten points on the base-line OT, what will be the velocity at that moment, and what will be the distance moved through by the stone, assuming the velocity uniform during the next interval, we obtain the following equations:

				FEET
At $t_0=0$	v_0	$32.2t=0$	$\Delta s_0=0 \times 0.25=$	0
$t_1=0.25$	v_1	$=8.05$	$\Delta s_1=8.05\times 0.25=$	2.0125
$t_2=0.50$	v_2	$=16.10$	$\Delta s_2=16.1\times 0.25=$	4.0250
$t_3=0.75$	v_3	$=24.15$	$\Delta s_3=24.15\times 0.25=$	6.0375
$t_4=1.00$	v_4	$=32.20$	$\Delta s_4=32.2\times 0.25=$	8.0500
$t_5=1.25$	v_5	$=40.25$	$\Delta s_5=40.25\times 0.25=$	10.0625
$t_6=1.50$	v_6	$=48.30$	$\Delta s_6=48.30\times 0.25=$	12.0750
$t_7=1.75$	v_7	$=56.35$	$\Delta s_7=55.35\times 0.25=$	14.0875
$t_8=2.00$	v_8	$=64.40$	$\Delta s_8=64.40\times 0.25=$	16.1000
$t_9=2.25$	v_9	$=72.45$	$\Delta s_9=72.45\times 0.25=$	18.1125

At the end of the interval commencing with t_9, the time $T = 2.5$ seconds, will have been reached, and the total computed fall, $s = 90.5625$ feet.

It will be evident that this result will necessarily be too small, owing to the manner in which the process has been conducted, since we have taken the initial velocity of the stone in each equation in order to obtain the distance traversed in the interval, thus ignoring the influence of acceleration during the interval. If, however, instead of taking the interval of time Δt as 1/4th second, we reduced it one-half, or made it 1/8th second, each equation of the type

$$\Delta s = v \Delta t$$

would be more nearly true, and the resulting sum 95.595 feet, would be also more nearly true, but there would be 20 equations to sum up, instead of 10. It is evident that as we take Δt smaller and smaller, we increase the number of equations, but we arrive closer to the truth.

Strictly speaking, therefore, we should make Δt indefinitely small or equal to dt, and sum an indefinitely great number of equations of the type

$$ds = vdt.$$

The sum of this indefinitely great number of equations would give s, accurately; or

$$s = \text{sum of all terms } vdt$$

taken from $t = 0$ to $t = 2.5$. This is expressed in the language of the integral calculus by

$$s = \int_0^T vdt,$$

where $\int$ stands for "the sum of all terms of the type." The superscript T, and the subscript 0, show that the upper and lower limits of the variable t, between which the summation is to be effected, are T seconds and 0 seconds, respectively.

The rules and theory of the integral

calculus show that the operation indicated by the integral $\int$, which it would be impossible to carry out arithmetically, since it is impossible to write down an infinite number of terms, is capable of being directly computed without performing any such process. In the case considered, the rules of the integral calculus show that

$$s = \frac{gT^2}{2}$$

or, that the integral

$$\int_0^T gt\,dt = \frac{gT^2}{2}.$$

Since

$$T = 2.5, \quad \frac{gT^2}{2} = \frac{32.2 \times 2.5 \times 2.5}{2}$$

$$= 100.625 \text{ feet.}$$

This would be the sum arrived at were it possible to write down and sum up the indefinitely great number of terms required in the case considered.

By reference to Fig. 8 it will be seen that the integral which we have considered is capable of a simple geometrical interpretation. For if we consider any interval of time $\varDelta t$ = 1/4th second say that following the time $t = 1$ second, the velocity at this time is 32.2 feet per second, represented by the ordinate $1v$. The space traversed at this velocity in the succeeding quarter second is

$$\varDelta s = v\varDelta t = 32.2 \times \frac{1}{4} = 8.05,$$

and is evidently the area of the vertical strip $1vca$, whose sides are v and $\varDelta t$, respectively. From this it will be seen that the total space traversed in 2.5 seconds, as summed from ten equations, is the sum of the areas of the vertical strips indicated in Fig. 8 by the dotted lines. The true distance traversed is, however, by the integral calculus,

$$s = \frac{1}{2} g T^2.$$

In the figure, the ordinate TV is equal to gT, and the base OT is equal to T, so that

$$s = \frac{1}{2} \times TV \times OT \qquad \text{units of area}$$

or, the area of the triangle $OTV = 100.625$.

It is evident from the figure that as we reduce the length of the intervals Δt, we make the vertical strips more numerous, and their total area will more nearly agree with the area of the whole triangle; and finally when the intervals dt are indefinitely small, and the steps indefinitely numerous, they will exactly conform at their terminations with the line $O v V$, and their aggregate area will exactly coincide with the area of the triangle OTV.

Any integral may, therefore, be regarded as the area of a certain geometrical

figure and to be composed of an indefinitely great number of elementary strips which, finally, coincide completely with the area.

An important practical application of the process of integration is found in many forms of meters. Consider, for example, a gas meter. At any moment when the gas jets in a building are lighted, there will be a certain flow of gas into the building through the meter. This flow may be expressed by the symbol f. Then in any brief interval of time Δt, the quantity of gas Δq, admitted to the building will be

$$\Delta q = f\Delta t$$

and, if the number of gas jets remained unaltered and burned steadily, the quantity of gas consumed in any given number of hours would be

$$q = ft,$$

but if a number of gas jets are turned on and off irregularly, or if the pressure in the mains varies, so that the flow is not uniform, then the total quantity of gas supplied to the house is not proportional to the time. At any moment, however, the following equation is true:

$$dq = fdt;$$

dt, being an indefinitely brief interval of time, and dq, the corresponding minute quantity of flow. If we sum all the equations which may be formed between the time $t = T_1$, and $t = T_2$, we obtain a perfectly accurate theoretical statement for the quantity of gas supplied; namely,

$$q = \int_{T_1}^{T_2} fdt.$$

The meter, if properly constructed, will perform this integration or summation mechanically, and will give between the

time T_1, and the time T_2, the value of the integral written on the right-hand side of this equation. Meters of this type are therefore, frequently called *integrating meters.* Of course their operation does not require the use of the integral calculus, but the integral calculus enables their operation to be very briefly and accurately stated in symbolical language, and if there is any known law connecting the flow with the time, the integral is capable of being evaluated arithmetically.

Fig. 9, represents the flow of gas through a meter, during different successive intervals. It will be seen that during the daytime the flow falls to zero or to a very small quantity. At night-time it rises to a maximum and varies according to the demand in the building. If the ordinates are laid off to a proper scale of flow, and the abscissæ to a proper scale of

time, the area of the curve between the ordinates T_1 and T_2, will give the integral

$$q = \int_{T_1}^{T_2} f dt.$$

which the meter mechanically registers.

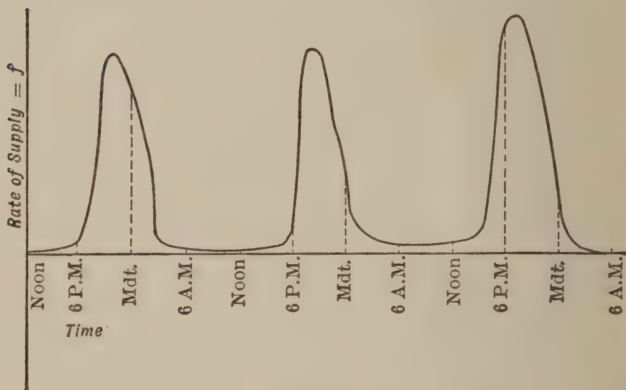

FIG. 9.

As another example of the meaning and application of the processes of the integral calculus, let us consider the determination of a height by means of barometric observations.

Let us suppose that at a certain station, say at the base of a mountain, and at some height h_1 metres above the mean sea level, the barometer shows an atmospheric pressure of p_1 millimetres of mercury. Then, from the known volume of air at the observed temperature and at this pressure p_1 the atmosphere behaves as though it were a layer of air of H_1 centimeters in height having a uniform density equal to the density at the point of observation, whereas, in reality, its density continually diminishes as we ascend, and its real height is consequently far greater. The height H_1 is called the height of the homogeneous atmosphere, or the virtual height to which the atmosphere would extend if its density remained constant at that observed.

If now we ascend with the barometer through a very small distance Δh meters, the virtual height H_1 of the atmosphere

above the original station will be reduced from H to $H - \varDelta H$, while the barometric pressure will be reduced from p to $p - \varDelta p$; $\varDelta p$, being the small fall in the barometric pressure due to the small difference in height $\varDelta h$. The proportional diminution in the virtual height of the atmosphere; or,

$$\frac{\varDelta H}{H} = \frac{\varDelta p}{p},$$

and since the diminution in virtual height $\varDelta H$, is the same as the change in real height $\varDelta h$, then

$$\frac{\varDelta h}{H} = -\frac{\varDelta p}{p}.$$

Multiplying both sides of this equation by H, we obtain

$$\varDelta h = -H\frac{\varDelta p}{p}.$$

It would be possible to call the new pressure p', at the slightly elevated or second station, which is $p - \varDelta p$, and repeat the equation for this new pressure and successively raise the barometer say a foot at a time, and write down a new equation of this type each time:

The elevation raised through

$$\varDelta h = -H\frac{\varDelta p}{p_{\mathrm{m}}},$$

where p_{m}, is the pressure at any station in the upward series. The total height through which the barometer had been raised from the first station to the last would then be found by summing up all these equations in the following manner:

$$\varDelta h_1 = -H\frac{\varDelta p}{p_1};$$

$$\varDelta h_2 = -H\frac{\varDelta p}{p_2};$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Delta h_n = -H\frac{\Delta p}{p'};$$

where p', stands for the pressure at the last or nth station. The sum of all the terms on the left-hand side of these equations is the total height $(h' - h)$ through which the barometer has been moved from the first station of elevation h, to the last station of which the required elevation is h'. The sum of the terms on the right-hand sides of these equations is the sum of all terms of the type $-H\frac{\Delta p}{p}$. The resulting summation equation will, however, not be quite correct, because each individual equation contains a small error due to the fact that in raising the barometer say a foot, the density of the air at the new elevation is somewhat less than that at the last preceding elevation, and the equation is therefore vitiated; but if

we suppose that, instead of raising the barometer a foot at a time, we raised it through an indefinitely small distance, and make an indefinitely great number of such stages of observation, each differing by dh metres, the imaginary equations become strictly accurate, and the sum total becomes strictly accurate, even although it would be practically impossible to perform the experiment in this way. But by the rules of the integral calculus we can determine what the sum of the indefinitely great number of terms on the right-hand side would be; for, it is expressed as the integral

$$h' - h = \int_{p}^{p'} - \frac{Hdp}{p}.$$

This integral is known to be

$$H \log_{\varepsilon} \frac{p}{p'},$$

and the total difference of elevation, or

$$(h' - h) = H \log_\varepsilon \frac{p}{p'},$$

or the total elevation, between the first and last station of the barometer is the product of H, the virtual height of the homogeneous atmosphere, and the Naperian logarithm of the ratio between the first and last pressures or readings of the barometer. Thus, if p = 750 millimetres, and p' = 375 millimetres of mercury, and H, the apparent height of the homogeneous atmosphere, 8×10^5 centimetres. Then

$$h = 8 \times 10^5 \log_\varepsilon \frac{30}{15};$$

$$8 \times 10^5 \log_\varepsilon 2 \qquad \text{cms.}$$

We may either look for the natural logarithm of the number 2 in tables of Naperian logarithms, or we may look for

the common logarithm of 2, and multiply it by the constant 2.3026 so that

$$\begin{aligned} h &= 8 \times 10^5 \times 2.3026 \times \log_{10} 2 \\ &= 8 \times 2.3026 \times 0.3010300 \times 10^5 \\ &= 5.546 \times 10^5 \text{ centimeters.} \\ &= 5{,}546 \times 10^3 \text{ metres.} \\ &= 5{,}546 \text{ metres or } 3.446 \text{ miles.} \end{aligned}$$

It is of course assumed either that the condition of the atmosphere remains uniform during the process of carrying the barometer up the mountain; or, that the two observations at the first and last stations are made simultaneously by two observers.

We have hitherto considered single integration. Just as it is possible, and often either necessary or convenient, in the differential calculus to employ a second differential co-efficient, or the differential of a differential, so, in the inverse operation of the integral calculus, it is often either neces-

sary or convenient to employ an integral of an integral, or a *double integral* as it is called, and which is indicated by the sign $\int\int$.

As an example of the natural introduction of a double integral, let us consider a water-pipe carrying a stream of water, and suppose that it be required to determine the total quantity of water q, which flows through a pipe in a given time t, from observations which determine the flow or velocity of movement of water, at different points of the cross-section of the pipe. In other words, we are supposed to know the velocity at different points of the cross-section and not to know the average velocity or total flow. If the velocity continued uniform at every point, it would only be necessary to find, from the velocity at each point, the average velocity, and this would give us at once the total

flow, because if A, be the area of cross-section of the pipe, in square centimetres, and v, the mean velocity in centimetres-per-second; then the volume of water flowing in each second will be

$V = Av$ cubic centimetres, or grammes, of water per second.

It is evident that a single integration will enable us to determine the average velocity v, from the assumed knowledge of the velocity at each point in the cross-section. But if we suppose that the velocity is not uniform, but varies according to an assigned law, not only at different points of the cross-section but also at different times, then we shall have to determine not only the average velocity of the cross-section at any moment by the single integration, but also the average velocity in time by a second integration,

and the solution of the problem may, therefore, be capable of expression as a double integration, in which one integration refers to space and the other to time.

When, therefore, such an equation is presented as the following

$$F = \int_{y_1}^{y_2}\int_{x_1}^{x_2} A\, dx\, dy,$$

it means that the quantity F, is a double integral, or the integral of an integral.

The equation may be represented as follows:

$$F = \int_{y_1}^{y_2} \left\{ \int_{x_1}^{x_2} A\, dx \right\} dy$$

or, representing the quantity inside the bracket by B,

$$= F\int_{y_1}^{y_2} B dy.$$

If, therefore, we integrate the quantity A, with respect to x alone, that is consider-

ing x, as an independent variable, and denote by B, the integral so obtained, and then integrate the quantity B, with respect to y, alone, considering y, the independent variable, the result of the second integration will give the quantity F.

In the same way triple, quadruple or higher integrals may be regarded as a succession of integrals, one being taken at a time. The area of a plane figure whose sides conform to a definite geometrical law can be usually expressed as a double integral, the first integral being taken with respect to x, a length parallel to one axis of co-ordinates, and the second integral being taken with respect to y, a length parallel to the other axis of co-ordinates. Similarly, the volume of a figure bounded by outlines which are defined by any geometrical law can usually be expressed as a triple integral, the first integral being

taken with respect to x, the second with respect to y, and the third with respect to z. In other words, the volume is expressed as equal to an indefinitely great number of little elements of volume, each of which has a cube of dimensions $dx \times dy \times dz$. Thus the equation

$$V = \int\int\int A\, dx\, dy\, dz$$

represents the simplest form of a volume or triple integral. In practice, triple integration is, usually, as far as multiple integration extends.

CHAPTER XIII.

DETERMINANTS.

The following pair of symmetrical equations,

$$\begin{aligned} 3x + y &= 5 \\ 2x + 3y &= 8, \end{aligned} \qquad (A)$$

are called *simultaneous equations* involving two unknown quantities; namely, x and y. From these two equations both of the unknown quantities may be computed.

Similarly, from the three following simultaneous equations,

$$\begin{aligned} 4x + 2y + 3z &= 17 \\ 7x + y - 2z &= 3 \\ x + 9y - 4z &= 7, \end{aligned} \qquad (B)$$

each of the three unknowns x, y, and z, can be computed; and, in general, any set of n, simultaneous equations, of the first degree, all of which are independent of each other; *i. e.*, are not reducible one to another, so that each contains an independent statement,—are sufficient for the determination of n unknown quantities.

When only two simultaneous equations have to be dealt with, as in the pair given above, the computation is a simple matter, and, in some cases, it can be carried out by mere inspection. It is easy to see that the pair of above equations are satisfied by the results,

$$x = 1 \; y = 2$$

since

$$3 \times 1 + 2 = 5$$
$$2 \times 1 + 3 \times 2 = 8.$$

When, however, a set of simultaneous

equations containing more than three unknowns is given, the computation, while not necessarily difficult, is often very tedious and lengthy. It has been found that the process may be conducted in a regular way, which can be formulated by a sort of algebraic shorthand, which enables the process to be carefully inspected, checked, and often simplified. This process has led to the introduction and use of what are called *determinants.*

A determinant consists of a symmetrical assemblage of quantities in rows and colums, bounded by a pair of vertical lines, there being as many rows as there are columns. Thus:

$$\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \qquad \begin{vmatrix} 2 & -r \\ q & 3 \end{vmatrix} \qquad \begin{vmatrix} a & g \\ b & h \end{vmatrix}$$

are determinants of the second order,

because each determinant has two rows and two columns. Similarly,

$$\begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} \qquad \begin{vmatrix} h & -o & w \\ q & r & o \\ s & a & t \end{vmatrix} \qquad \begin{vmatrix} -6 & 4 & z \\ a & 0 & 0 \\ 3 & 5 & 7 \end{vmatrix}$$

are determinants of the third order, because each has three rows and three columns. Again,

$$\begin{vmatrix} 16 & 4 & 8 & 2 & 25 \\ 3 & 22 & 20 & 11 & 9 \\ 15 & 6 & 4 & 23 & 17 \\ 24 & 18 & 12 & 10 & 1 \\ 7 & 5 & 21 & 19 & 13 \end{vmatrix}$$

is a determinant of the fifth order, because it has five rows and five columns.

The separate numbers, or symbols, in a

determinant; or, as they are called, the *elements*, are to be multiplied according to a definite rule. A determinant of the second order naturally produces terms or products having two elements each. A determinant of the third order naturally produces terms or products of three elements each; and a determinant of the nth order naturally produces terms or products of n elements, each. Thus:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - gec - hfa - idb,$$

or the determinant is identical with the product:

$$aei + dhc + gbf - gec - hfa - idb.$$

This determinant of the third order is, therefore, a brief method of expressing

these six terms, each containing products of three elements. The terms are obtained by taking one element in each horizontal row, and combining with it one letter in each of all the other rows. Thus, the first term *aei*, takes the first element of the first row, the second element of the second row, and the third element of the third row, affixing to the same the positive sign. The fifth term has the first letter of the first row, the third letter of the second row, and the second letter of the third row, prefixing to the product the negative sign. In this way each term has one element, and only one element, out of each row, and takes the positive or negative sign according to the way in which the selection is made, following a definite rule.

A *complete determinant; i. e.*, a determinant which has no zeros in it, of the 2d order, is identically equivalent to two

terms, each consisting of the product of two elements.

A complete determinant of the third order is identically equivalent to six terms, each consisting of the product of three elements.

A complete determinant of the fourth order is identically equivalent to twenty-four terms, each consisting of the product of four elements.

A complete determinant of the fifth order is identically equivalent to 120 terms, each consisting of the product of five elements.

A complete determinant of the nth order is equivalent to $n!$ terms, each consisting of the product of n elements.

The application of determinants may be illustrated sufficiently for our present purposes by considering the three simultaneous equations (B).

The value of the unknown quantity x, in these three equations, is obtained by the quotient of two determinants; namely,

$$x = \frac{\begin{vmatrix} 17 & 2 & 3 \\ 3 & 1 & -2 \\ 7 & 9 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 3 \\ 7 & 1 & -2 \\ 1 & 9 & -4 \end{vmatrix}} = \frac{294}{294} = 1$$

Similarly, the value of y, is the ratio of another pair of determinants:

$$y = \frac{\begin{vmatrix} 4 & 7 & 3 \\ 7 & 3 & -2 \\ 1 & 7 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 3 \\ 7 & 1 & -2 \\ 1 & 9 & -4 \end{vmatrix}} = \frac{588}{294} = 2$$

and, finally, the value of z, is the ratio of a third set-of determinants ; namely,

$$z = \frac{\begin{vmatrix} 4 & 2 & 17 \\ 7 & 1 & 3 \\ 1 & 9 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 3 \\ 7 & 1 & -2 \\ 1 & 9 & -4 \end{vmatrix}} = \frac{882}{294} = 3.$$

Each of the determinants in the numerator and denominator may be worked out by the equation already given for a determinant of the 3d order. The student will observe that the determinants forming the denominators of the above fractions are all the same, and that the determinants forming the numerators differ from each other in a manner which is readily traced by reference to the coefficients in the three simultaneous equations (B).

There are various rules for treating, simplifying and reducing determinants, but it will be sufficient for the student to remember that a determinant is an abbreviated form of a symmetrical set of products such as commonly presents itself in the process of solving simultaneous equations.

CHAPTER XIV.

SYNOPSIS OF SYMBOLS COMMONLY FOUND IN MATHEMATICAL FORMULÆ.

$+$ Plus, or sign of addition. $5 + 7 = 12$.

$-$ Minus, or sign of subtraction. $7 - 5 = 2$.

$\pm$ Plus or minus. $7 \pm 5 = 12$ or 2.

$\sim$ Difference sign. $5 \sim 7 = +2$. The difference between 5 and 7 is 2.

$>$ Greater than. $7 > 5$; seven is greater than five.

$<$ Less than. $5 < 7$; five is less than seven.

$=$ Equality. $5 + 7 = 12$.

$\neq$ Inequality. $5 \neq 7$; five is not equal to seven: or, 7 and 5 are unequal. (2).

$\equiv$ Identity. $2(a+b) \equiv 2a+2b$; both members invariably identically equal.

$\geqq$ Equality or Superiority. $f \geqq 30$; f greater than or equal to 30.

$\leqq$ Equality or Inferiority. $f \leqq 30$; f less than or equal to 30.

$\sim$ Nearly equal to. $0.667 \sim \frac{2}{3}$.

$\times$ Multiplication. $5 \times 7 = 35$.

. Multiplication. $a \,.\, b = a \times b = ab$.

$\div$ Division. $3 \div 4 = 0.75$.

— Division bar. $\dfrac{3}{4} = 3 \div 4 = 0.75$.

/ Division solidus. $3/4 = 3 \div 4 = 0.75$.

( ) { } [ ] Brackets or Parentheses. $5(6+7) = 5\{6+7\} = 5[6+7] = 5 \times 13$.

——— Vinculum line. $5 \times \overline{6+7+8} = 5(6+7+8) = 5 \times 21 = 105$.

∞, Infinity. $3 \times 3 \times 3 \ldots\ldots$ *ad infinitum* $= \infty$.

$\propto$, Varies as. Pressure of a liquid column $\propto$ the depth of liquid.

$\therefore$, Therefore. Because $3 + 5 = 8$ $\therefore$ $3 + 3 + 5 = 3 + 8$.

$\because$, Since. $3 + 3 + 5 = 3 + 8$ $\because$ $3 + 5 = 8$.

$: :: :$, Proportionality. $3 : 5 :: 6 : 10$; three is to five, as is six to ten.

^{2}, Square. $3^2 = 9$.

^{3}, Cube. $3^3 = 27$.

^{n}, Index or Exponent. $3^n = 3 \times 3 \times 3$, n times in all.

$^{-n}$, Negative Index. $3^{-n} = \frac{1}{3^n}$.

$^{\frac{1}{2}}$, Surd Exponent. $4^{\frac{1}{2}}$ = square root of $4 = 2$.

$^{\frac{1}{n}}$, Fractional Exponent. $4^{\frac{1}{n}}$ = nth root of 4.

$\sqrt{}$ or $\sqrt[2]{}$, Radical or Surd. $\sqrt{4} = 4^{\frac{1}{2}} = 2$ = square root of 4.

$\sqrt[3]{}$, Cube Root. $\sqrt[3]{27} = 3$; $\sqrt[3]{729} = 9$.

$\sqrt[n]{\ }$, nth Root. $\sqrt[n]{a} = a^{\frac{1}{n}} = n$th root of a.

! or $\llcorner$, Factorial. $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

Σ, Summation. $\Sigma(abc)$ = sum of all terms of the type abc.

$\left|\ \right|$, Determinant. $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = a \times d - b \times c$.

π, Pi (Greek). Ratio $\dfrac{\text{Circumference}}{\text{Diameter}} = 3.14159 \ldots.$

$\log_{10} x$, Common Logarithm of x. $\log_{10} 100 = 2.000$.

$\log_\epsilon x$, or hyp. log. x, Naperian Logarithm of x. $\log_\epsilon 100 = 4.6052$.

i or j, Sign of the imaginary. $i = j = \sqrt{-1}$.

$f(x)$ Function of x. x^2, $\sqrt{2}$, $\log x$, ax, are functions of x denotable by $f(x)$.

$\sin \alpha$, Sine of angle α. Perpendicular $\div$ hypothenuse.

cos α, Cosine of angle α. Base ÷ hypothenuse.

tan α, Tangent of angle α. Perpendicular ÷ base.

cot α, Cotangent of angle α. Reciprocal of tangent.

sec α, Secant of angle α. Reciprocal of cosine.

cosec α, Cosecant of angle α. Reciprocal of sine.

vers α, Versed sine of angle α. $1 - \cos \alpha$.

$\sin^{-1} \alpha$, Inverse sine. The angle whose sine is *alpha*.

sinh α, Hyperbolic sine of angle α.

cosh α, Hyperbolic cosine of angle α.

tanh α, Hyperbolic tangent of angle α. $\sinh \alpha / \cosh \alpha$.

coth α, Hyperbolic cotangent of angle α. $1/\tanh \alpha$.

sech α, Hyperbolic secant of angle α. $1/\cosh \alpha$.

cosech α, Hyperbolic cosecant of angle α. $1/\sinh \alpha$.

Δy, Difference of y.

dy, Differential of y. Limit of Δy.

$\frac{dy}{dx}$, Differential coefficient of y, with respect to x. Limiting ratio of $\frac{\Delta y}{\Delta x}$, when $\Delta x = 0$.

$\frac{d^2y}{dx^2}$, Second differential coefficient of y, with respect to x. $\frac{d\left(\frac{dy}{dx}\right)}{dx}$.

$\frac{d^ny}{dx^n}$, nth differential coefficient of y to x.

$\dot{y}$, Differential of y, with respect to time. $\frac{dy}{dt}$.

y', Differential of y, with respect to space. $\frac{dy}{ds}$.

$\int$, Integration sign.

$\int f(x)dx$, Integral of $f(x)$, with respect to x.

$\int\int$, Double Integral.

$\int\int\int$, Triple Integral.

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